I Year B.E/B.Tech, Degree Examination

Key to FAQ

in

Engineering Mathematics - I to II

SCHOOL OF DISTANCE EDUCATION

ANDHRA UNIVERSITY

VISAKHAPATNAM - 530 003

Key to FAQ in Engineering Maths

Dear Learner

Some of the students attending PCP have expressed difficulties in answering Engineering Mathematics Papers and asked for some solved examples. Accordingly, key to frequently asked questions in Engineering Mathematics (Old Question Papers of SDE) has been prepared. You are advised to workout some more problems from other books also like Engineering Mathematics. Vol-1 by P.P Gupta, Krishna Prakashan Media Pvt. Ltd., & A text book on Engineering Mathematics by Bali & Iyengar. Laxmi Publications Pvt. Ltd.

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B.E / B.Tech. Ist Year

MATHEMATICS - I

(Common for all branches)

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1. (a) If $z = x\, f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$**, prove that**

$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0.$

Sol:- Let $u = x\, f\left(\frac{y}{x}\right),\ v = g\left(\frac{y}{x}\right)$

Then $z = u + v$ ____(1)

u is a homogeneous function of degree 1.

v is a homogeneous function of degree 0.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1(u) = u \qquad \text{____(2)}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0(v) = 0. \qquad \text{____(3)}$$

(2) + (3) gives,

$$x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = u + 0$$

or $$x\frac{\partial}{\partial x}(u+v) + y\frac{\partial}{\partial y}(u+v) = u.$$

or $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=u \qquad \text{\_\_\_\_(4)}$$

Differentiating (4) partially with respect to 'x' we get

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial u}{\partial x} \qquad \text{\_\_\_\_(5)}$$

Differentiating (4) partially with respect to 'y' we get

$$x\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=\frac{\partial u}{\partial y} \qquad \text{\_\_\_\_(6)}$$

x (5) + y (6) gives

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

$$\Rightarrow\ x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+u=u \quad \text{(using 2 \& 4)}$$

$$\boxed{\Rightarrow\ x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0}$$

1. (b) Transform the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ **into polar co-ordinates.**

Sol:- We have $x=r\cos\theta,\ y=r\sin\theta$ and $r=\sqrt{\left(x^2+y^2\right)}$,

$$r=\sqrt{\left(x^2+y^2\right)}, \theta=\tan^{-1}\left(y/x\right) \quad \frac{\partial r}{\partial x}=\frac{x}{\sqrt{\left(x^2+y^2\right)}}=\cos\theta$$

and $\dfrac{\partial\theta}{\partial x}=-\dfrac{y}{x^2+y^2}=-\dfrac{\sin\theta}{r}$

Thus $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial r}.\dfrac{\partial r}{\partial x}+\dfrac{\partial u}{\partial\theta}.\dfrac{\partial\theta}{\partial x}=\cos\theta\dfrac{\partial u}{\partial r}-\dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial\theta}$

i.e. $\dfrac{\partial}{\partial x}=\cos\theta\dfrac{\partial}{\partial r}-\dfrac{\sin\theta}{r}\dfrac{\partial}{\partial\theta}$

Similarly $\dfrac{\partial}{\partial y}=\sin\theta\dfrac{\partial}{\partial r}+\dfrac{\cos\theta}{r}\dfrac{\partial}{\partial\theta}$.

$$\therefore \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}$$

$$+\frac{\partial^2 u}{\partial\theta^2}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} \quad .....(i)$$

and $\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)\left(\sin\theta\dfrac{\partial}{\partial r}+\dfrac{\cos\theta}{r}\dfrac{\partial}{\partial\theta}\right)$

$$\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\sin^2\theta\ \frac{\partial^2 u}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial r}\qquad ......\text{(ii)}$$

Adding (i) and (ii), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}.$$

Hence the transformed equation is

$$\boxed{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0}.$$

2. (a) If $x^2+y^2+z^2-2xyz=1$ **show that**

$$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}+\frac{dz}{\sqrt{1-z^2}}=0.$$

Sol:- Given $x^2+y^2+z^2-2xyz=1$ ____(1)

Taking differentials we get,

$$2x\,dx+2y\,dy+2z\,dz-2y\,zdx-2x\,zdy-2xy\,dz=0$$

or $(x-yz)dx+(y-zx)dy+(z-xy)dz=0$ ____(2)

Now $(x-yz)^2\ =x^2-2xyz+y^2z^2$

$$= 1 - y^2 - z^2 + y^2 z^2 \qquad \text{using (1)}$$

$$= \left(1 - y^2\right) - z^2\left(1 - y^2\right)$$

$$= \left(1 - y^2\right)\left(1 - z^2\right)$$

$$\Rightarrow \ (x - yz) \ = \ \sqrt{\left(1 - y^2\right)\left(1 - z^2\right)}$$

Similarly

$$(y - zx) = \sqrt{\left(1 - z^2\right)\left(1 - x^2\right)}$$

$$(z - xy) \ = \ \sqrt{\left(1 - x^2\right)\left(1 - y^2\right)}$$

Substituting in (2), we get

$$\sqrt{\left(1 - y^2\right)\left(1 - z^2\right)}\ dx + \sqrt{\left(1 - z^2\right)\left(1 - x^2\right)}\ dy$$

$$+ \sqrt{\left(1 - x^2\right)\left(1 - y^2\right)}\ dz = 0$$

or
$$\boxed{\frac{dx}{\sqrt{1 - x^2}} + \frac{dy}{\sqrt{1 - y^2}} + \frac{dz}{\sqrt{1 - z^2}} = 0}$$

2. (b) If $|\alpha| < |$ show that $\int_0^{\pi} \frac{\log(1 + \alpha \cos x)}{\cos x} dx = \pi \sin^{-1}\alpha.$

Sol:- Let $F(\alpha) \ = \ \int_0^{\pi} \frac{\log(1 + \alpha \cos x)}{\cos x} dx$

then $F'(\alpha) = \int_0^{\pi} \frac{\partial}{\partial\alpha}\left(\frac{\log(1+\alpha\cos x)}{\cos x}\right)dx$

$$= \int_0^{\pi} \frac{1}{(1+\alpha\cos x)}\,dx$$

$$= \int_0^{\pi} \frac{1}{\left(\cos^2\frac{x}{2}\sin^2\frac{x}{2}\right)+\alpha\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}dx$$

$$= \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{(1+\alpha)+(1-\alpha)\tan^2\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t,\ \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$F'(\alpha) = \int_0^{\infty} \frac{2}{(1+\alpha)+(1-\alpha)t^2}\,dt$$

$$= \frac{2}{1-\alpha}\int_0^{\infty} \frac{1}{\left(\frac{1+\alpha}{1-\alpha}\right)+t^2}\,dt$$

$$= \frac{2}{(1-\alpha)}\frac{\sqrt{1-\alpha}}{\sqrt{1+\alpha}}\left[\tan^{-1}\frac{t\sqrt{1-\alpha}}{\sqrt{1+\alpha}}\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{1-\alpha^2}}\left(\frac{\pi}{2}\right) = \frac{\pi}{\sqrt{1-\alpha^2}}$$

$$F'(\alpha) = \frac{\pi}{\sqrt{1-\alpha^2}}$$

Integrating w.r.t 'α'

$$F(\alpha) = \pi \sin^{-1}\alpha + c$$

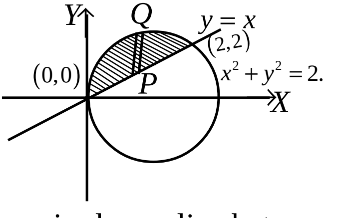
$$F(0) = 0 \Rightarrow c = 0$$

$$\boxed{\therefore\ F(\alpha) = \pi \sin^{-1}\alpha}$$

3. (a) Find the area enclosed by the curves $y^2 = 4x - x^2$ and $y = x$ in the fist quadrant, using double integration.

Sol:- The equation of the circle is $x^2 + y^2 - 4x = 0$ and the line is $y = x$.

The Curves intersect at the points whose abscissae are given by $4x - x^2 = x^2$



or $2x^2 = 4x$

or $x = 0, 2.$

Using vertical strips, the required area lies between $y = x,\ y = \sqrt{4x - x^2}$ *and* $x = 0, x = 2.$

$$\therefore \quad \text{Required area} = \int_{x=0}^{2} \int_{x}^{\sqrt{4x-x^2}} dy\,dx$$

$$= \int_{x=0}^{2} \left(\sqrt{4x - x^2} - x\right) dx$$

$$= \int_{x=0}^{2} \left(\sqrt{4-(x-2)^2} - x\right) dx$$

$$= \left[\frac{(x-2)}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^2 - \left(\frac{x^2}{2}\right)_0^2$$

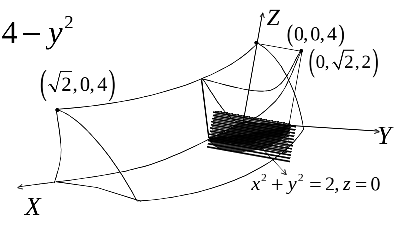
$$= -2\sin^{-1}(-1) - 2$$

$$= \boxed{(\pi - 2)}$$

3. b) Find the volume bounded by the paraboloid $z = 2x^2 + y^2$ and the parabolic cylinder $z = 4 - y^2$.

Sol:- The intersection of the two surfaces is given by solving

$z = 2x^2 + y^2$ and $z = 4 - y^2$



$z = 4 - y^2$

$2x^2 + y^2 = 4 - y^2$

$\Rightarrow \quad 2x^2 + 2y^2 = 4 \ or \ x^2 + y^2 = 2$

$\therefore$ The projection of the required region on the xy-plane is a circular region given by, $x^2 + y^2 = 2, \ z = 0$

Consider the volume in the first octant. The projection of this part on the xy - plane is quadrant of the circle $x^2 + y = 2, \ z = 0.$

$\therefore$ Required volume $=$ 4 (Volume of the region in the first octant)

$$= 4 \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{z=2x^2+y^2}^{4-y^2} dz\,dy\,dx$$

$$= 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} (z)_{2x^2+y^2}^{4-y^2}\,dydx.$$

$$= 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} (4-2y^2-2x^2)dydx$$

$$= 4\int_0^{\sqrt{2}}\left[\left(4-2x^2\right)y-\frac{2}{3}y^3\right]_0^{\sqrt{2-x^2}} dx.$$

$$= 4\int_0^{\sqrt{2}}\left[2\left(2-x^2\right)\sqrt{2-x^2}-\frac{2}{3}\left(2-x^2\right)^{\frac{3}{2}}\right]dx$$

$$= 4\left(\frac{4}{3}\right)\int_0^{\sqrt{2}}\left(2-x^2\right)^{\frac{3}{2}} dx$$

Put $x = \sqrt{2}\sin\theta$

$$dx = \sqrt{2}\cos\theta\ d\theta$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{2}} 2^{3/2}.\cos^3\theta.2^{1/2}\cos\theta d\theta$$

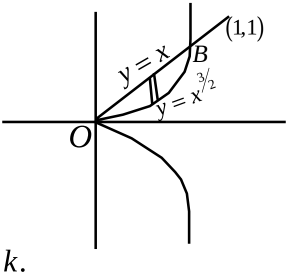
$$= \frac{64}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\ d\theta$$

$$= \boxed{\frac{64}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = 4\pi.}$$

4. a) Find the centre of gravity of the area in the first quadrant lying between the curves.

Sol:- Let $(\bar{x}, \bar{y})$, be the C.G of the given region.

$$\bar{x} = \frac{1}{M} \iint_A \rho x dx dy$$



$$\bar{y} = \frac{1}{M} \iint_A \rho y dx dy$$

Here ρ is not given $\therefore$ let $\rho = k$.

Now $M = \iint_A \rho dx dy$. Where 'A' is the region shown in the figure.

The points of intersection of $y=x$ and $y^2 = x^3$ are given

$y^2 = y^3 \Rightarrow y = 0,1.$

$\therefore$ 0 (0,0) and B (1,1) are the points of intersection.

$$M = \iint_A \rho dx dy = \int_{x=0}^{1} \int_{y=x^{3/2}}^{x} k dy dx = k\int_0^1 (y)_{x^{3/2}}^{x} dx$$

$$= k\int_0^1 \left(x - x^{3/2}\right) dx$$

$$= k\left(\frac{x^2}{2} - \frac{2}{5}x^{5/2}\right)_0^1 = k\left(\frac{1}{2} - \frac{2}{5}\right) = \frac{k}{10}.$$

$$\therefore \bar{x} = \frac{1}{M}\int_0^1 \int_{x^{3/2}}^{x} kx dy dx.$$

$$= \frac{k}{M}\int_0^1 x\left(y\right)_{x^{3/2}}^{x} dx = \frac{k}{M}\int_0^1 x\left(x - x^{3/2}\right)dx$$

$$= \frac{k}{M}\int_0^1 \left(x^2 - x^{5/2}\right)dx$$

$$= \frac{k}{M}\left(\frac{x^3}{3} - \frac{2}{7}x^{7/2}\right)_0^1 = \frac{k}{M}\cdot\frac{1}{21} = \frac{10}{21}.$$

Similarly

$$\bar{y} = \frac{1}{M}\int_0^1 \int_{y=x^{3/2}}^{x} kydydx = \frac{k}{M}\int_0^1 \left(\frac{y^2}{2}\right)_{x^{3/2}}^{x} dx$$

$$= \frac{k}{2M}\int_0^1 \left(x^2 - x^3\right)dx = \frac{k}{2M}\left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{k}{2M}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{k}{24M} = \frac{10}{24} = \frac{5}{12}$$

Hence the required C.G $\boxed{= \left(\frac{10}{21}, \frac{5}{12}\right)}$

4. b) Evaluate $\int_0^2 x^4\left(8 - x^3\right)^{-\frac{1}{3}} dx$ **in terms of Beta function.**

Sol:- Consider $I = \int_0^2 x^4\left(8 - x^3\right)^{-\frac{1}{3}} dx$.

Put $x^3 = 8t.\ or\ x = 2t^{1/3}$.

$$3x^2 dx = 8\,dt$$

$$I = \int_0^1 2^4 . t^{4/3} . 8^{-1/3} . (1-t)^{-1/3} . \frac{8dt}{(3).(4)t^{2/3}}$$

$$= \frac{16.8}{12.2}\int_0^1 t^{2/3}(1-t)^{-1/3}\,dt = \frac{16}{3}\int_0^1 t^{-\frac{5}{3}-1}(1-t)^{\frac{2}{3}-1}\,dt$$

$$= \boxed{\frac{16}{3}\,\beta\left(\frac{5}{3},\frac{2}{3}\right)}$$

5. a) Find the co-ordinates of the foot of the perpendicular from the origion on the line given by

$x + 2y + 3z + 4 = 0 = x + y + z + 1.$

Sol:- Let the line of intersection of the given planes be AB.

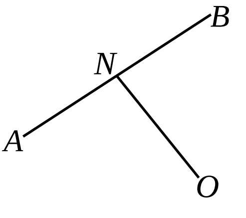
Let N be the foot of the perpendicular form the origion on AB.

Let 1, m, n be the d.c's of AB, then we have

$$1 + 2m + 3m = 0 \qquad \_\_\_\_(1)$$

$$1 + m + n = 0 \qquad \_\_\_\_(2)$$

Solving (1) and (2) we get



$$\frac{l}{2-3} = \frac{m}{3-1} = \frac{n}{1-2}$$

$$l : m : n = -1 : 2 : -1.$$

$\therefore$ The d.r.'s of AB are -1, 2, -1.

Let the given line AB pass through the plane z=0, then putting z=0 in the given planes, we get

$$x+2y+4=0 \qquad \text{\_\_\_\_(3)}$$

$$x+y+1=0 \qquad \text{\_\_\_\_(4)}$$

Solving (3) and (4) we get $x=2, y=-3.$

Hence the line AB passes through the point (2, -3, 0). Therefore equations of AB are,

$$\frac{x-2}{-1}=\frac{y+3}{2}=\frac{z-0}{-1}\left(=r\ say\right).$$

Any point on this line is $\left(-r+2,\ 2r-3,\ -r\right)$

Let this point be 'N' where ON is perpendicular to AB and O is the origin.

Direction ratio's of ON are $\left(-r+2,\ 2r-3,\ -r\right)$

Lines ON and AB are perpendicular

$$\therefore \left(-r+2\right)\left(-1\right)+\left(2r-3\right)\left(2\right)+\left(-r\right)\left(-1\right)=0.$$

$$\Rightarrow\ r=\frac{4}{3}.$$

Thus coordinates of 'N' are

$$\boxed{\left(\frac{-4}{3}+2, 2.\frac{4}{3}-3, -\frac{4}{3}\right) i.e. \left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)}$$

5. b) Find the equation of the plane parallel to the line $x-2=\frac{y-1}{3}=\frac{z-3}{2}$ which contains the points (-3, 1, 2) and (0, 0, 0).

Sol:- The general equation of the plane through (-3, 1, 2) is

$a(x+3)+b(y-1)+c(z-2)=0$ ____(1)

If (0, 0, 0) lies on this plane, then we have

$3a-b-2c=0$ ____(2)

The plane (1) is parallel to the line $\frac{x-2}{1}=\frac{y-1}{3}=\frac{z-3}{2}$

$\therefore$ we have $a+3b+2c=0$ ____(3)

Solving (2) & (3) we get

$$\frac{a}{-2+6}=\frac{b}{-2-6}=\frac{c}{9+1}$$

$\therefore a=4k,\ b=-8k,\ c=10k.$

Substituting in (1), we get the required equation of the plane as

$$4k(x+3)-8k(y-1)+10k(z-2)=0$$

or $4(x+3)-8(y-1)+10(z-2)=0$

or $\boxed{2x-4y+5z=0.}$

6. a) **Find the length and equation of the shortest distance between the lines** $2x+y-z=0=x-y+2z$; $x+2y-3z-4=0=2x-3y+4z-5.$

Sol:- Let the given two straight lines be L_1, L_2

$L_1 : 2x+y-z=0=x-y+2z$ ____(1)

$L_2 : x+2y-3z-4=0=2x-3y+4z-5$ ____(2)

First we reduce the equations of the lines to the symmetrical form.

To find the d.r.'s of L_1.

Let l_1, m_1, n_1 be the d.c.'s of L_1. Since it lies on both planes, we have

$$2l_1 + m_1 - n_1 = 0$$

$$l_1 - m_1 + 2n_1 = 0$$

$$\Rightarrow \frac{l_1}{2-1} = \frac{m_1}{-1-4} = \frac{n_1}{-2-1}$$

or $$\frac{l_1}{1} = \frac{m_1}{-5} = \frac{n_1}{-3}.$$

$\therefore$ The d.r.'s if the line L_1 are 1, -5, -3 ____(3)

To fnd one point on L_1,

Put $z=0$ in (1), we get

$$\left.\begin{aligned} 2x+y=0 \\ x-y=0 \end{aligned}\right\} \Rightarrow \; x=0,\; y=0.$$

$\therefore$ One point L_1 is (0, 0, 0).

$\therefore$ From (2) & (3) the equations of the line L_1 in symmetrical form are

$$\frac{x}{1}=\frac{y}{-5}=\frac{z}{-3} \qquad \text{\_\_\_\_(4)}$$

to fuid the d.r.s' of L_2.

Let l_1, m_2, n_2 be the d.c.s' of L_2 then we have

$$\left.\begin{aligned} l_2+2m_2-3n_2=0 \\ 2l_2-3m_2+4n_2=0 \end{aligned}\right\}$$

$$\Rightarrow \quad \frac{l_2}{8-9}=\frac{m_2}{-6-4}=\frac{n_2}{-3-4}$$

$$\text{or} \quad \frac{l_2}{-1}=\frac{m_2}{-10}=\frac{n_2}{-7}$$

$\therefore$ The d.r.s'of L_2 are (-1, -10, -7) ____(6)

To find one point on L_2, putting z=0 in (2) we get,

$$\left.\begin{aligned} x+2y=4 \\ 2x-3y=5 \end{aligned}\right\} \Rightarrow \; x=\frac{22}{7}$$

$$y = \frac{3}{7}$$

$\therefore$ One point L_2 is $\left(\frac{22}{7}, \frac{3}{7}, 0\right)$

Form (6) & (7) the equations of the line L_2 in symmetrical

form are $\dfrac{x - \frac{22}{7}}{-1} = \dfrac{y - \frac{3}{7}}{-10} = \dfrac{z - 0}{-7}$ ____(7)

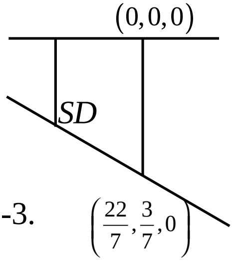
Let l, m, n be the d.c.'s of the S.D between (1) and (2) since S.D is perpendicular to both the lines

(1) and (2) $\quad l - 5m - 3n = 0$

$$-l - 10m - 7n = 0$$

$$\Rightarrow \frac{l}{35-30} = \frac{m}{3+7} = \frac{n}{-10-5}$$

$$\frac{l}{5} = \frac{m}{10} = \frac{n}{-15}$$



or $\quad \dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{-3}.$

$\therefore$ D.cs of S.D are proportional to 1,2,-3.

$\therefore$ Actual d.cs are $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}.$

Now one point online (1) is (0,0,0)

and one point on line (2) is $\left(\frac{22}{7}, \frac{3}{7}, 0\right)$

$$= \frac{1}{\sqrt{14}}\left(\frac{22}{7} - 0\right) + \frac{2}{\sqrt{14}}\left(\frac{3}{7} - 0\right)\frac{-3}{\sqrt{14}}(0)$$

$$= \frac{(22+6)}{7\sqrt{14}} \quad = \frac{4}{\sqrt{14}}.$$

The S.D is the line of intersection of the two planes

(1) Containing the line (1) and the S.D

(2) Containing the line (2) and the S.D

$\therefore$ its equations are $\begin{vmatrix} x & y & z \\ 1 & -5 & -3 \\ 1 & 2 & -3 \end{vmatrix} = 0$

and $\begin{vmatrix} x - 22/7 & y - 3/7 & z/-7 \\ -1 & -10 & -7 \\ 1 & 2 & -3 \end{vmatrix} = 0.$

6. b) Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$.

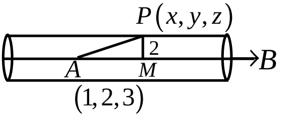
Sol:- Let AB be the axis of the cylinder whose equations are

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2} \quad \text{\_\_\_\_(1)}$$

Its d.r's are 2,1,2

Dividing each by $\sqrt{4+1+4}=3$

its d.c's are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$.



Let $P(x, y, z)$ by any point on the cylinder.

Join AP and draw PM perpendicular on AB so that

MP = radius of the cylinder = 2

From $\Delta\ AMP \quad AP^2 = AM^2 + MP^2$

$$= AM^2 + 4. \qquad \_\_\_\_(2)$$

Also $AP^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$

and $AM =$ Projection of AP on the axis AB.

$$= \frac{2}{3}(x-1) + \frac{1}{3}(y-2) + \frac{2}{3}(z-3)$$

$$= \frac{1}{3}(2x + y + 2z - 10)$$

$\therefore$ From (2),

$$(x+1)^2 + (y-2)^2 + (z-3)^2$$

$$= \frac{(2x + y + 2z - 10)^2}{9} + 4$$

or $9\left[(x-1)^2+(y-2)^2+(z-3)^2\right]$

$$=\left[(2x+y+2z)-10\right]^2+36$$

or $5x^2+8y^2+5z^2-4yz-8zx$

$$-4xy+22x-16y-14z-10=0$$

Which is the required equation of the cylinder.

7. a) Discuss the convergence of the geometic series $1+x+x^2+.....\infty$ •

Sol:- The Geometric series

i) Converges if $-1<x<1$

ii) Diverges if $x\geq 1$

iii) Oscillates finitely if $x=-1$

iv) Oscillates infinitely if $x<-1$

i) $-1<x<1 \; i.e \; |x|<1$

Let $S_m=1+x+x^2+......+x^{n-1}$

$$=\frac{1(1-x^n)}{1-x}=\frac{1}{1-x}-\frac{x^n}{1-x}$$

$$\therefore \underset{n\to\infty}{Lt} S_n=\frac{1}{1-x}.\left(\therefore \underset{n\to\infty}{Lt} x^n=0 \, as |x|<1\right)$$

$\Rightarrow \{S_n\}$ is convergent, hence the given series converges.

ii) $x \geq 1$ *i.e,* $x = 1$ *and* $x > 1$

For $x = 1,\ S_n = 1 + 1 + 1 + \ldots\ldots\ldots$ to n terms

$= n$

$$\therefore \lim_{n \to \infty} S_n = \infty$$

$\Rightarrow \{S_n\}$ is divergent, hence the given series is also divergent,

For $x > 1$,

$$S_n = 1 + x + \ldots. + x^{n-1}$$

$$= \frac{1 - x^n}{1 - x}$$

$$\therefore \lim_{n \to \infty} S_n = \infty \qquad \left(\because \lim_{n \to \infty} x^n = \infty,\ as\, x > 1\right)$$

$\Rightarrow \{S_n\}$ is divergent, hence the given series is also divergent.

iii) $x = -1$

$$S_n = 1 - 1 + 1 - 1 + \ldots.\ to\ n\ terms$$

$=$ 1 or 0 if n is odd or even

$$\therefore \lim_{n \to \infty} S_n = 1\, or\, 0$$

$\Rightarrow \{S_n\}$ oscillates finitely and hence the given series oscillates finitely.

iv) $x<-1 \Rightarrow -x>1$

Let $p=-x$ then $p>1$

Now $S_n = 1+x+....+x^{n-1}$

$$= \frac{1-x^n}{1-x} = \frac{1-(-p)^n}{1+p} = \frac{1+p^n}{1+p} \text{ is n is odd}$$

$$= \frac{1-p^n}{1+p} \text{ is n is even}$$

$\therefore \underset{n\to\infty}{Lt} S_n = -\infty \ or \ +\infty$

$\therefore \{S_n\}$ Oscillates inifinitely and hence the given series oscillates infinitely.

7. b) Test the convergence of the following series:

i) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + ...\infty.$

ii) $\sum_{n=1}^{\infty} \frac{1.4.7...(3n-2)}{2.5.8...(3n-1)}.$

Sol:- i) Here $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$\therefore \underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left(\frac{n+1}{n+2}\right)\left(\frac{n}{n+1}\right)^{1/2} x^2$$

$$= \underset{n\to\infty}{Lt}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\left(\frac{1}{1+\frac{1}{n}}\right)^{1/2} x^2$$

$$= x^2$$

$\sum u_n$ converges if $x^2 < 1$ and diverges if $x^2 > 1$.

If $x^2 = 1$, Ratio test fails.

$$u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

Let $v_n = \frac{1}{n^{3/2}}$

$\therefore$ By comparison test, $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)} = 1 \neq 0$

$\therefore \sum v_n$ is convergent, $\therefore \sum u_n$ is convergent.

Hence the given series is convergent for $x^2 \leq 1$ and is divergent for $x^2 > 1$.

ii) Sol :-

Here $u_n = \frac{1.4.7\ .....\ (3n-2)}{2.5.8.....(3n-1)}$

$$u_{n+1} = \frac{1.4.7\,.....\,(3n-2)(3n+1)}{2.5.8.....(3n-1)(3n+2)}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left(\frac{3n+1}{3n+2}\right) = \underset{n\to\infty}{Lt}\left(\frac{\left(1+\frac{1}{3n}\right)}{\left(1+\frac{2}{3n}\right)}\right) = 1.$$

Ratio test fails.

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{3n+2}{3n+1}-1\right] = n\left(\frac{1}{3n+1}\right) = \frac{1}{\left(3+\frac{1}{n}\right)}$$

$$\underset{n\to\infty}{Lt}\,n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{1}{3} < 1.$$

$\therefore$ **By Raabe's test the series diverges.**

8. a) State and prove Cauchy's Root test

Sol :- Statement: If $\sum u_n$ is a positive term series and $\underset{n\to\infty}{Lt}(u_n)^{1/n} = \lambda,$ then the series is (i) Convergent if $\lambda < 1$ (ii) divergent if $\lambda > 1$ and test fails if $\lambda = 1$.

Let $\lambda \geq 0,$ for $\lambda < \frac{\lambda+1}{2} < 1,$ there exists a natural number l such that $0 < (u_n)^{1/n} < \frac{\lambda+1}{2}$ for all $n \geq l.$ Thus

$u_n < \left(\frac{\lambda+1}{2}\right)^n$ for all $n \geq 1$. Since $\sum\left(\frac{\lambda+1}{2}\right)^n$ is convergent for $0 < \frac{\lambda+1}{2} < 1$.

By comparison test $\sum u_n$ is convergent.

For $\lambda > 1$, there exists a natural number l such that $(u_n)^{1/n} > 1$

$\Rightarrow u_n > 1$ for all $n \geq l$

$\Rightarrow \underset{n\to\infty}{Lt}\, u_n \neq 0$

$\Rightarrow \sum u_n$ diverges.

If $(u_n)^{1/n} \to +\infty$, there exists a natural number l such that $(u_n)^{1/n} > 1$ i.e., $u_n > 1$ for $n \geq l$.

$\Rightarrow \underset{n\to\infty}{Lt}\, u_n \neq 0$

$\Rightarrow \sum u_n$ diverges.

For $\lambda = 1$, we note that $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n}$ have $\underset{n\to\infty}{Lt}\,(u_n)^{1/n} = 1$ but the first series is convergent and the second series is divergent.

$\therefore$ **The test fails if** $\lambda = 1$.

8. b) Show that the series $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ converges uniformly on [a,1], 0<a<1 but not on [0,1].

Sol:- On $[a,1]$ where $0<a<1$,

$$|u_n(x)| = \left|\frac{x}{1+n^2x^2}\right| \le \frac{1}{1+n^2a^2}$$

and now, to test the convergence of $\sum \frac{1}{(1+n^2a^2)}$.

Let $f_n = \frac{1}{1+n^2a^2}$, $f_{n+1} = \frac{1}{1+(n+1)^2 a^2}$

$$\underset{n\to\infty}{Lt}\left(\frac{f_{n+1}}{f_n}\right) = \underset{n\to\infty}{Lt} \frac{1+n^2a^2}{1+(n+1)^2 a^2}$$

$$= \underset{n\to\infty}{Lt}\left(\frac{\frac{1}{n^2}+a^2}{\frac{1}{n^2}+\left(1+\frac{1}{n}\right)^2 a^2}\right)$$

$$= \frac{a^2}{a^2} = 1.$$

$\therefore$ 'D' Alenbert's ratio test fails.

Now, we shall apply Raabe's test

$$n\left(\frac{f_n}{f_{n+1}} - 1\right) = n\left(\frac{1+(n+1)^2 a^2}{1+n^2a^2} - 1\right)$$

$$= n\left(\frac{2na^2 + a^2}{1+n^2a^2}\right)$$

$$\underset{n\to\infty}{Lt}\left(\frac{f_n}{f_{n+1}} - 1\right) = \underset{n\to\infty}{Lt}\frac{n.n\left(2a^2 + \frac{a^2}{n}\right)}{n^2\left(\frac{1}{n^2} + a^2\right)}$$

$$= \frac{2a^2}{a^2} = 2 > 1.$$

$\therefore \sum f_n$ converges.

$\therefore$ By Weierstrass's M - test, the given series Converges uniformly on [a, 1]

Let the given series be uniformly convergent on [0,1], then for $\varepsilon = \frac{1}{8} > 0$ then extists m, such that

$$\left|\frac{x}{1+m^2x^2} + \frac{x}{1+(m+1)^2 x^2} + ... + \frac{x}{1+(2m)^2 x^2}\right| < \frac{1}{8}$$

$$\therefore \left|\frac{mx}{1+(2m)^2 x^2}\right| < \frac{1}{8} \text{ (by taking } n = m\text{)}$$

On putting $x = \frac{1}{m}$ it gives $\frac{1}{5} < \frac{1}{8}$, contradiction, therefore the given series is not uniformly convergent on [0,1].

9. a) Obtain the Fourier series of $f(x) = x + x^2 I(-\pi, \pi)$.

Hence prove that $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:- Let $f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ____(1)

Then $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx$

$$= \frac{2}{\pi} \int_{0}^{\pi} x^2 dx$$

$$= \frac{2}{\pi} . \frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos nx \, dx \qquad \left(\therefore \int_{-\pi}^{\pi} x \cos nx dx = 0 \right)$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - (2x) \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{2\pi}{n^2} \cos n\pi \right] = \frac{4}{n^2} \cos n\pi$$

$$= \frac{4}{n^2} (-1)^n .$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx dx$$

$$\left(\therefore \int_{-\pi}^{\pi} x^2 \sin nx dx = 0\right)$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{-2(-1)^n}{n}$$

Substituting the values of a_0, a_n, b_n in (1) we get

$$\boxed{f(x) = \frac{\pi^2}{3} + \sum_{1}^{\infty}\left(\frac{4}{n^2}(-1)^n \cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right)}.$$

9. b) Expand $f(x) = x^2, 0 < x < 4$ with period $T = 4$ in a Fourier series.

Sol:- Let $f(x) = \frac{a}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

Here $2l = 4 \Rightarrow l = 2$

$$\therefore \quad x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$$

Then $a_0 = \frac{1}{2}\int_0^4 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^4 = \frac{32}{3}$

$$a_n = \frac{1}{2}\int_0^4 x^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[x^2\left(\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right) - 2x\left(\frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right)\right.$$

$$\left. + 2.\frac{-8}{n^3\pi^3}\sin\left(\frac{n\pi x}{2}\right)\right]_0^4$$

$$= \frac{1}{2}\left[\frac{2x^2}{n\pi}\sin\frac{xn\pi}{2} + \frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}\right]_0^4$$

$$= \frac{1}{2}\left[\frac{32}{n^2\pi^2}\cos 2n\pi\right] = \frac{16}{n^2\pi^2}$$

$$b_n = \frac{1}{2}\int_0^4 x^2 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[x^2\left(\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right) - 2x\left(\frac{-4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right)\right.$$

$$\left. + 2.\frac{8}{n^3\pi^3}\cos\left(\frac{n\pi x}{2}\right)\right]_0^4$$

$$= \frac{1}{2}\left[\frac{-2x^2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{8x}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right) + \frac{16}{n^3\pi^3}\cos\left(\frac{n\pi x}{2}\right)\right]_0^4$$

$$= \frac{1}{2}\left[\frac{-32}{n\pi}\cos 2n\pi\right] = \frac{-16}{n\pi}$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$\boxed{x^2 = \frac{16}{3} + \frac{16}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}\cos\left(\frac{n\pi x}{2}\right) - \frac{1}{n}\sin\left(\frac{n\pi x}{2}\right)\right]}$$

10. a) Find the Fourier Cosine series of $f(x) = \cos ax,$ $0 < x < \pi$ **given that a is not an integer.**

Sol:- Let us extend the function $f(x)$ in the interval $-\pi < x < 0$ so tha the new function becomes even function in the interval $-\pi < x < \pi.$

Hence the Fourier series of $f(x)$ over the interval $(-\pi, \pi)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} \cos ax dx = \frac{2}{\pi}\left[\frac{\sin ax}{a}\right]_0^{\pi} = \frac{2}{a\pi}\sin a\pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos ax.\cos nx dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\cos(n+a)x + \cos(n-a)x\right]dx$$

$$=\frac{1}{\pi}\left[\frac{\sin(n+a)x}{(n+a)}+\frac{\sin(n-a)x}{(n-a)}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\frac{1}{\left(n^2-a^2\right)}\left[2n\sin nx\cos ax-2a\cos nx\sin ax\right]_0^{\pi}$$

$$=\frac{1}{\pi\left(n^2-a^2\right)}(-2a\cos n\pi.\sin a\pi)=\frac{2a\sin a\pi}{\pi\left(n^2-a^2\right)}(-1)^{n+1}$$

Hence the required Fourier series is

$$\cos\ ax=\frac{\sin a\pi}{a\pi}+\frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\left(n^2-a^2\right)}\cos nx$$

10. b) Using the Fourier Series expansion of $f(x)=|x|$ in $(-\pi,\pi)$, show that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}$.

Sol:- Since $f(-x)=|-x|=|x|=f(x)$

$\therefore\ f(x)$ is an even function and hence $b_n=0.$

Let $f(x)=|x|=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx.$

Then $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}|x|dx=\frac{2}{\pi}\int_0^{\pi}x\,dx$

$$=\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}=\pi.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left(\cos n\pi - 1\right)$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right] = 0 \quad \text{if } n \text{ is even}$$

$$= \frac{-4}{n^2\pi} \quad \text{if } n \text{ is odd}$$

$$\therefore |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}(\cos nx)$$

Using Parseval's identity

$$\int_0^{\pi} |x|^2\,dx = \frac{\pi}{2}\left[\frac{a_0^2}{2} + \sum_1^{\infty} a_n^2\right]$$

$$\frac{\pi^3}{3} = \frac{\pi}{2}\left[\frac{\pi^2}{2} + \sum_1^{\infty}\frac{16}{\pi^2(2n-1)^4}\right]$$

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\frac{\pi^2}{12} \times \frac{\pi^2}{8} = \sum_{1}^{\infty}\frac{1}{(2n-1)^4}.$$

$$\boxed{\frac{\pi^4}{96} = \sum_{1}^{\infty}\frac{1}{(2n-1)^4}.}$$

11. (a) If v be a function of r, where $r^2 = x^2 + y^2$, show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2}$

Sol:- Given $v = f(r)$, $r^2 = x^2 + y^2$

Consider $r^2 = x^2 + y^2$ ____(1)

Differentiating partially w.r.t 'x' we get $2r\frac{\partial r}{\partial x} = 2x$

or $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly differentiating (1) Partially w.r.t 'y' we get

$\frac{\partial r}{\partial y} = \frac{y}{r}.$

Now $v = f(r)$

$$\therefore \ \frac{\partial v}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\ \frac{x}{r}$$

$$= \frac{\partial v}{\partial r}\ \frac{x}{r} \qquad\qquad (\because v = f(r))$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{r}\ \frac{\partial v}{\partial r} + x\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right)\frac{\partial v}{\partial r} + \frac{x}{r}\ \frac{\partial^2 v}{\partial r^2}\ \frac{\partial r}{\partial x}$$

$$= \frac{1}{r}\ \frac{\partial v}{\partial r} - \frac{x}{r^2}\ .\ \frac{x}{r}\frac{\partial v}{\partial r} + \frac{x}{r}\ \frac{\partial^2 v}{\partial r^2}\ .\ \frac{x}{r}$$

$$= \frac{1}{r}\ \frac{\partial v}{\partial r} - \frac{x^2}{r^3}\frac{\partial v}{\partial r} + \frac{x^2}{r^2}\ \frac{\partial^2 v}{\partial r^2}$$

$$= \left(\frac{r^2 - x^2}{r^3}\right)\frac{\partial v}{\partial r} + \frac{x^2}{r^2}\ \frac{\partial^2 v}{\partial r^2}$$

$$= \frac{y^2}{r^3}\ \frac{\partial v}{\partial r} + \frac{x^2}{r^2}\ \frac{\partial^2 v}{\partial r^2} \qquad \text{\_\_\_\_\_(2)}$$

Similarly $\dfrac{\partial^2 v}{\partial y^2} = \dfrac{x^2}{r^3}\ \dfrac{\partial v}{\partial r} + \dfrac{y^2}{r^2}\ \dfrac{\partial^2 v}{\partial r^2}$ _____(3)

$$\therefore \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \left(\frac{x^2 + y^2}{r^3}\right)\frac{\partial v}{\partial r} + \frac{\left(x^2 + y^2\right)}{r^2}\ \frac{\partial^2 v}{\partial r^2}$$

$$= \frac{1}{r}\ \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2}.$$

or $$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r} f'(r) + f''(r)}$$

11. (b) If $u = 2xy,\ v = x^2 - y^2,\ x = r\ \cos\theta,\ y = r\ \sin\theta$ **Compute** $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Sol:- Given $u = 2xy,\ v = x^2 - y^2,\ x = r\cos\theta,\ y = r\sin\theta$

$$u = 2.r\cos\theta\ .\ r\ \sin\theta$$

$$= 2r^2\cos\theta\ \sin\theta = r^2\ \sin 2\theta.$$

$$v = r^2\cos^2\theta - r^2\sin^2\theta = r^2\cos 2\theta.$$

$$\therefore \quad \frac{\partial u}{\partial r} = 2r\sin 2\theta,\ \frac{\partial u}{\partial\theta} = 2\ r^2\cos 2\theta$$

$$\frac{\partial v}{\partial r} = 2r\cos 2\theta,\ \frac{\partial v}{\partial\theta} = -2\ r^2\sin 2\theta.$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial\theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial\theta} \end{vmatrix} = \begin{vmatrix} 2r\sin 2\theta & 2r^2\cos 2\theta \\ 2r\cos 2\theta & -2r^2\sin 2\theta \end{vmatrix}$$

$$= -4r^3\left(\sin^2 2\theta + \cos^2 2\theta\right)$$

$$\boxed{= -4r^3}$$

12. (a) Obtain Taylor's expansion of $\tan^{-1}\left(\frac{y}{x}\right)$ about (1,1) upto and including the second degree terms.

Sol:- $f(x,y)=\tan^{-1}\frac{y}{x}$, $f(1,1)=\tan^{-1}1=\frac{\pi}{4}$

$$f_x(x,y)=\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2},\ f_x(1,1)=-\frac{1}{2}$$

$$f_y(x,y)=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=\frac{x}{x^2+y^2},\ f_y(1,1)=\frac{1}{2}$$

$$f_{xx}(x,y)=-y(-1)\left(x^2+y^2\right)^{-2}.2x=\frac{2xy}{\left(x^2+y^2\right)^2},$$

$$f_{xx}(1,1)=\frac{1}{2}$$

$$f_{xy}(x,y)=\frac{\left(x^2+y^2\right).1-x.2x}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2},$$

$$f_{xy}(1,1)=0$$

$$f_{yy}(x,y)=x(-1)\left(x^2+y^2\right)^{-2}.2y=-\frac{2xy}{\left(x^2+y^2\right)^2},$$

$$f_{yy}(1,1)=-\frac{1}{2}$$

$$\therefore \quad \tan^{-1}\frac{y}{x} = f(x,y)$$

$$= f(1,1)+\left[(x-1)\, f_x(1,1)+(y-1)\, f_y(1,1)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2\, f_{xx}(1,1)\right.$$

$$\left.+2(x-1)(y-1)\, f_{xy}(1,1)+(y-1)^2\, f_{yy}(1,1)\right]+.....$$

$$= \frac{\pi}{4}+\left[(x-1).\left(-\frac{1}{2}\right)+(y-1)\frac{1}{2}\right]$$

$$+\frac{1}{2!}\left[(x-1)^2.\frac{1}{2}+2(x-1)(y-1).0+(y-1)^2.\left(-\frac{1}{2}\right)\right]+.....$$

$$\boxed{=\frac{\pi}{4}-\frac{1}{2}(x-1)+\frac{1}{2}(y-1)+\frac{1}{4}(x-1)^2-\frac{1}{4}(y-1)^2+....}$$

12. (b) Find the shortest and the longest distances from the point $(1,2-1)$ **to the sphere** $x^2+y^2+z^2=24.$

Sol:- Let (x,y,z) be any point on the sphere. Distance of the point $A(1,2-1)$ from (x,y,z) is given by

$$\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$$

If the distance is maximum or minimum, so will be the square of the distance.

Let $f(x,y,z) = (x-1)^2 + (y-2)^2 + (z+1)^2$ ____(1)

$\phi(x,y,z) = x^2 + y^2 + z^2 = 24$ ____(2)

Consider Lagrange's function

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$= (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda\left(x^2 + y^2 + z^2 - 24\right)$$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2(x-1) + \lambda 2x = 0$

or $(x-1) + \lambda x = 0$ ____(3)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2(y-2) + \lambda 2y = 0$

or $(y-2) + \lambda y = 0$ ____(4)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2(z+1) + \lambda 2z = 0$

or $(z+1) + \lambda z = 0$ ____(5)

(3), (4), (5) can be written as

$$x-1 = -\lambda x$$

$$y-2 = -\lambda y$$

$$z+1 = -\lambda z$$

$$\Rightarrow \quad \frac{x-1}{y-2} = \frac{x}{y} \qquad \Rightarrow \qquad y = 2x.$$

$$\text{and} \quad \frac{y-2}{z+1} = \frac{y}{z} \qquad \Rightarrow \qquad -2z = y$$

$$z = -\frac{y}{2} = -x.$$

Substituting in (2) we get

$$x^2 + 4x^2 + x^2 = 24$$

$$\Rightarrow \quad 6x^2 = 24 \qquad \Rightarrow \qquad x = \pm 2.$$

Thus we get two points $P(2,4,-2)$, $Q(-2,-4,\ 2)$ on the sphere which are at maximum or minimum distances from the given point 'A'.

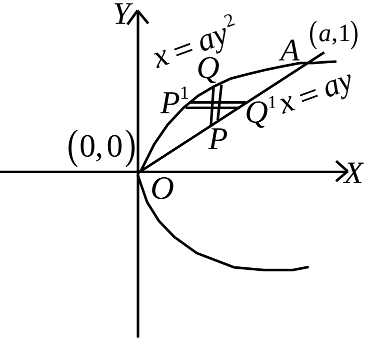
$$\text{Now} \ AP = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2} = \sqrt{6}$$

$$AQ = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2} = \sqrt{54}$$

$\therefore$ $P(2,4,-2)$ is at a minimum distance from A and the minimum distance is $\sqrt{6}$

$Q(-2,-4,2)$ is at a maximum distance from A and the maximum distance is $\sqrt{54}$.

13. (a) Evaluate $\int_0^a \int_{x/a}^{\sqrt{x/a}} \left(x^2+y^2\right) dx\,dy$ by changing the order of integration.



Sol:- From the limits of integration, it is clear that we have to integrate first w.r.t 'y' from $y = x/a$ to $y = \sqrt{x/a}$ and then w.r.t x from x=0 to x=a.

Thus integration is first performed along the vertical strip PQ which extends from a point P on the line $x = ay$ to the point Q on the parabola $x = ay^2$. Then the strip is sliding from 'O' to $A(a,1)$, the point of intersection of the two curves.

For changing the order of integration, we divide the region into horizontal strips P^1Q^1 which extends from P^1 on the parabola $x = ay^2$ to Q^1 on the line $x = ay$.

$$\therefore \int_0^a \int_{x/a}^{\sqrt{x/a}} \left(x^2+y^2\right) dx\,dy = \int_{y=0}^{a} \int_{x=ay^2}^{ay} \left(x^2+y^2\right) dx\,dy$$

$$= \int_0^1 \left(\frac{x^3}{3} + y^2 x\right)_{ay^2}^{ay} dy$$

$$= \int_0^1 \left(\frac{a^3 y^3}{3} + ay^3 - \frac{a^3 y^6}{3} - ay^4 \right) dy$$

$$= \left[\frac{a^3 y^4}{12} + \frac{ay^4}{4} - \frac{a^3 y^7}{21} - \frac{ay^5}{5} \right]_0^1$$

$$= a^3 \left(\frac{1}{12} - \frac{1}{21} \right) + a \left(\frac{1}{4} - \frac{1}{5} \right)$$

$$\boxed{= \frac{a^3}{28} + \frac{a}{20}.}$$

13. (b) Evaluate $\iiint \left(lx + my + mz^2 \right) dx\, dy\, dz$ **taken through out the sphere** $x^2 + y^2 + z^2 \le a^2$.

Sol:- Changing to sphercial polar coordinates,we have $x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta,$

$dx\, dy\, dz = r^2 \sin\theta\ dr\, d\theta\, d\phi.$

Equation of the sphere is $r = a$.

$$\iiint_V \left(lx + my + mz \right)^2 dx\, dy\, dz$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} \left(l\, r \sin\theta \cos\phi + m\, r \sin\theta \sin\phi + n\, r \cos\theta \right)^2 r^2 \sin\theta\, dr\, d\theta\, d\phi\,.$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{0}^{a} \left(l \sin\theta \cos\phi + m \sin\theta \sin\phi + n \cos\theta \right)^2 r^4 \sin\theta\ dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \Big[\left(l\cos\phi + m\sin\phi \right)^2 .$$

$$\sin^2\theta + n^2 \cos^2\theta + 2n\cos\theta . \sin\theta \left(l\cos\phi + m\sin\phi \right) \Big] .$$

$$\sin\theta \, . \left(\frac{r^5}{5} \right)_0^a \, d\theta \, d\phi$$

$$= \frac{a^5}{5} \int_{\phi=0}^{2\pi} \int_0^{\pi} \Big[\left(l\cos\phi + m\sin\phi \right)^2 \sin^3\theta + n^2 \cos^2\theta \, \sin\theta \, +$$

$$2n\sin^2\theta . \cos\theta . \left(l\cos\phi + m\sin\phi \right) d\theta \, d\phi \Big]$$

$$= \frac{a^5}{5} \int_0^{2\pi} \left[2 \int_0^{\pi/2} \left(l\cos\phi + m\sin\phi \right)^2 \sin^3\theta \, d\theta \right.$$

$$\left. + \, 2 \int_0^{\pi/2} n^2 \cos^2\theta \sin\theta \, d\theta + 0 \right] d\phi$$

$$= \frac{a^5}{5} \int_0^{2\pi} \left[\not{2} \left(l\cos\phi + m\sin\phi \right)^2 . \frac{\Gamma 2 \, \Gamma \frac{1}{2}}{\not{2} \, \Gamma \frac{5}{2}} + 2n^2 . \frac{\Gamma \frac{3}{2} \Gamma 1}{2 \Gamma \frac{5}{2}} \right] d\phi .$$

$$= \frac{a^5}{5} \int_0^{2\pi} \left[\frac{4}{3} \left(l\cos\phi + m\sin\phi \right)^2 + n^2 . \frac{2}{3} \right] d\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left[\frac{4}{3}\left\{l^2\cos^2\phi + m^2\sin^2\phi\right.\right.$$

$$\left.\left. + 2lm\cos\phi\,\sin\phi\right\} + \frac{2n^2}{3}\right]d\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left[\frac{4}{3}\left\{l^2\left(\frac{1+\cos 2\phi}{2}\right) + \right.\right.$$

$$\left.\left. m^2\left(\frac{1-\cos 2\phi}{2}\right) + lm\sin 2\phi\right\} + \frac{2n^2}{3}\right]d\phi$$

$$= \frac{a^5}{5}\left[\frac{4}{3}\frac{\left(l^2+m^2\right)}{\cancel{2}}.\cancel{2}\pi + \frac{4}{3}.\left(\frac{l^2-m^2}{2}\right)\left(\frac{\sin 2\phi}{2}\right)_0^{2\pi}\right.$$

$$\left. + \ lm\left(\frac{-\cos 2\phi}{2}\right)_0^{2\pi} + \frac{2n^2}{3}.2\pi\right]$$

$$= \frac{a^5}{5}\left[\frac{4}{3}\pi\left(l^2+m^2+n^2\right)\right]$$

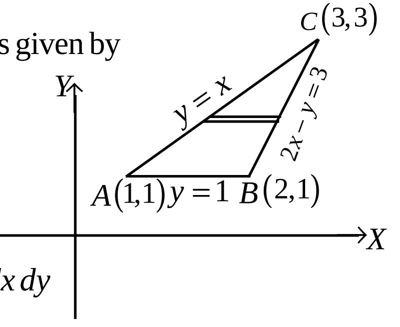
$$= \boxed{\frac{4}{15}\pi a^5\left(l^2+m^2+n^2\right)}.$$

14. (a) Determine the M.I about the X-axis of the area of a triangle with the vertices A (1,1), B (2,1) and C (3,3).

Sol:- Here the region R is the ΔABC whcih is bounded by $y = 1, y = x \ and \ 2x - y = 3.$

Let $P = K$.

M.I about the X-axis is given by



$$I_x = \iint_R P y^2 \, dx \, dy$$

$$= K \int_{y=1}^{3} \int_{x=y}^{3+y/2} y^2 \, dx \, dy$$

$$= K \int_1^3 y^2 \left(x\right)_y^{3+y/2} dy$$

$$= K \int_1^3 y^2 \left[\frac{3+y}{2} - y\right] dy$$

$$= \frac{K}{2} \int_1^3 y^2 (3-y) \, dy$$

$$= \frac{K}{2} \left[\cancel{3}\frac{y^2}{\cancel{3}} - \frac{y^4}{4}\right]_1^3$$

$$= \frac{K}{2} \left[27 - \frac{81}{4} - 1 + \frac{1}{4}\right]$$

$$= \frac{K}{2}(6) = 3K.$$

Mass of the lamina

$$M = \iint_R P\,dx\,dy$$

$$= K \int_{y=1}^{3} \int_{x=y}^{3+y/2} dx\,dy.$$

$$= K \int_1^3 \left(\frac{3+y}{2} - y \right) dy = \frac{K}{2} \int_1^3 (3-y)\,dy.$$

$$= \frac{K}{2} \left[3y - \frac{y^2}{2} \right]_1^3 = \frac{K}{2} \left[9 - \frac{9}{2} - 3 + \frac{1}{2} \right]$$

$$= \frac{K}{2} [6-4] = K\,.$$

$$\therefore \quad \boxed{I_x = 3K = 3M}$$

14. (b) Prove that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{B(m,n)}{a^n b^m}$ where $m, n,$ a, b are positive.

Sol:- Put $bx = at$

or $x = \frac{at}{b}$ so that $dx = \frac{a}{b}\, dt.$

$$\therefore \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \int_0^\infty \frac{\left(\frac{at}{b} \right)^{m-1}}{(a+at)^{m+n}} \cdot \left(\frac{a}{b} \right) dt$$

$$= \frac{1}{a^n b^m} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \boxed{\frac{B(m,n)}{a^n \, b^m}}.$$

15. (a) Find the equations to the line that intersects the lines $x+y+z=1,\ 2x-y-z=2,\ x-y-z=3,\ 2x+4y-z=4$ **and passes through the point (1,1,1).**

Sol:- Equations of the line intersecting the given lines are

$$\left.\begin{array}{l} (x+y+z-1)+\lambda_1(2x-y-z-2)=0 \\ (x-y-z-3)+\lambda_2(2x+4y-z-4)=0 \end{array}\right\} \quad \text{——(1)}$$

This line passes through (1,1,1)

$\therefore$ We must have $(1+1+1-1)+\lambda_1(2-1-1-2)=0$

$$\Rightarrow \lambda_1 = 1$$

$$(1-1-1-3)+\lambda_2(2+4-1-4)=0$$

$$\Rightarrow \lambda_2 = 4$$

Substituting for λ_1 and λ_2 in (1) we get

$x-1=0,\ 9x+15y-5z-19=0.$

These are the equations of the required line.

15. (b) Find the image of point P (1,3,4) in the plane $2x - y + z + 3 = 0$.

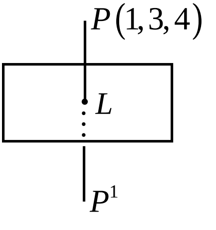
Sol:- The given plane is $2x - y + z + 3 = 0$ d.r.s of the normal to the plane are $2, -1, 1$.

$\therefore$ Equations of the line PL through P(1,3,4) and perpendicular to the plane (1) are $\dfrac{x-1}{2} = \dfrac{y-3}{-1} = \dfrac{z-4}{1}$

$(= r\ say)$.

$\therefore$ Co-ordinates of any point on this line are

$P^1(2r+1, -r+3, r+4)$ ______(2)



Where $PL = LP^1$ and L is the foot of the perpendicular form P on the plane.

Since L is the midpoint of PP^1

$\therefore$ its Co-ordinates are $\left(\dfrac{2r+1+1}{2}, \dfrac{-r+3+3}{2}, \dfrac{r+4+4}{2}\right)$

$$\left(r+1, \frac{6-r}{2}, \frac{8+r}{2}\right) \quad \text{\_\_\_\_\_\_(3)}$$

Since L lies on the plane (1),

$$2(r+1) - \left(\frac{6-r}{2}\right) + \left(\frac{8+r}{2}\right) + 3 = 0$$

$$\Rightarrow \quad r = -2$$

Putting this value of r in (2), we get the image of P as $P^1(-3,5,2)$.

16. (a) Find the equation of the sphere having its centre on the plane $4x-5y-z=3$ and passing through the circle

$x^2+y^2+z^2-2x-3y+4z+8=0, \quad x-2y+z=8.$

Sol:- The equation of the sphere can be taken as

$$\left(x^2+y^2+z^2-2x-3y+4z+8\right)+\lambda\left(x-2y+z-8\right)=0$$

$$\Rightarrow x^2+y^2+z^2+(\lambda-2)x-(3+2\lambda)y+(\lambda+4)z+8-8\lambda=0 \qquad \text{\_\_\_\_(1)}$$

The centre of this sphere is

$$=\left(\frac{2-\lambda}{2}, \frac{3+2\lambda}{2}, \frac{-\lambda-4}{2}\right)$$

which lies on the plane $4x-5y-z=3.$

$$\Rightarrow \quad 4\left(\frac{2\lambda}{2}\right)-5\left(\frac{3+2\lambda}{2}\right)-\left(\frac{-\lambda-4}{2}\right)=3$$

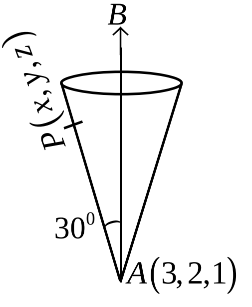
$$\Rightarrow \quad \lambda=\frac{-9}{13}.$$

Hence the equation of the required sphere is

$$13\left(x^2+y^2+z^2\right)+(-9-26)x-(39-18)y$$
$$+(-9+52)z+(104+72)=0$$

$$\Rightarrow \boxed{13\left(x^2+y^2+z^2\right)-35x-21y+43z+176=0}.$$

16. (b) Find the equation of the right circular cone whose vertex is (3,2,1), semi-vertical angel 30^0 and axis is the line $\frac{x-3}{4}=\frac{y-2}{1}=\frac{z-1}{3}$.



Sol:- Let $P(x,y,z)$ be any point on the cone. AB is the axis of the cone whose d.r.'s are 4,1,3. It passes through the vertex A (3,2,1).

$\angle PAB = 30^0$, the semi vertical angle.

d.r.'s of AP are $x-3,\ y-2,\ z-1$.

d.r.'s of AB are 4,1,3.

$$\therefore \cos 30^0 = \frac{4(x-3)+1(y-2)+3(z-1)}{\sqrt{16+1+9}\ \sqrt{(x-3)^2+(y-2)^2+(z-1)^2}}$$

$$\text{or } \left(\frac{3}{4}\right)26\left[(x-3)^2+(y-2)^2+(z-1)^2\right]$$

$$=\boxed{\left[4(x-3)+(y-2)+3(z-1)\right]^2}$$

17. (a) Show that if a series $\sum u_n$ is convergent then $\underset{n\to\infty}{Lt}\ u_n = 0.$

Sol:- Let S_n denote the n^{th} partial sum of the series $\sum u_n$. Then $\sum u_n$ is convergent.

$\Rightarrow \{S_n\}$ is convergent

$\therefore \underset{n\to\infty}{Lt}\ S_n = S$ (finite)

Also $\underset{n\to\infty}{Lt}\ S_{n-1} = S$

But $u_n = S_n - S_{n-1}$

$$\therefore \underset{n\to\infty}{Lt}\ u_n = \underset{n\to\infty}{Lt}\left(S_n - S_{n-1}\right)$$

$$= \underset{n\to\infty}{Lt}\ S_n - \underset{n\to\infty}{Lt}\ S_{n-1}$$

$$= \boxed{S - S = 0.}$$

17. (b) (i) Test for convergence of the following series

$$\frac{\sqrt{2}-\sqrt{1}}{1} + \frac{\sqrt{3}-\sqrt{2}}{2} + \frac{\sqrt{4}-\sqrt{3}}{3} + \ldots\ldots$$

Sol:- Here

$$u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{n\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$= \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$= \frac{1}{n\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$= \frac{1}{n^{3/2}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

Let $v_n = \frac{1}{n^{3/2}}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{1}{\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$= 1/2 \neq 0 \text{ (finite)}$$

$\therefore \sum v_n$ is convergent ($\sum \frac{1}{n^P}$, $P > 1$ for convergent)

$\therefore \sum u_n$ is also convergent by comparison test.

(ii) $\sum_{2}^{\infty} \frac{1}{n \log n}$.

Sol:- Here $u_n = \frac{1}{n \log n} = f(n)$

For $x \geq 1$, $f(x)$ is positive and monotonic decreasing.

$\therefore$ Integral test is applicable.

Now $\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x\log x}\,dx$

$= \underset{K\to\infty}{Lt} \int_2^{K} \frac{1}{x\log x}\,dx$

$= \underset{K\to\infty}{Lt} \left[\log(\log x)\right]_2^K$

$= \underset{K\to\infty}{Lt} \left[\log(\log K) - \log(\log 2)\right] \neq$ finite number

$\therefore \int_2^{\infty} f(x)\,dx$ is divergent.

By integral test $\sum u_n, \int_K^{\infty} f(x)\,dx$ for $x \geq 1$ converge or diverge together.

$\therefore \sum u_n$ **is divergent**.

18. (a) Show that the harmonic series of order $P, \sum_1^{\infty} \frac{1}{n^p}$. **converges for p>1 and diverges for 0<p<1.**

Sol:- Here $u_n = \frac{1}{n^p} = f(n)$ for $x \geq 1$, $f(x)$ is positive, and monotonic decreasing.

$\therefore$ integral test is applicable.

$\therefore$ The above series will converge or diverge accroding to

$\int_1^{\infty} \frac{dx}{x^p}$ is finite for infinite.

Case.1 $\quad P \neq 1$

$$\int_1^\infty \frac{dx}{x^p} = \underset{K\to\infty}{Lt} \int_1^K \frac{dx}{x^p} = \underset{K\to\infty}{Lt} \left(\frac{K^{1-P} - 1}{1-P} \right)$$

$$= \frac{1}{P-1}(finite), \; for\; P > 1$$

$$= \infty \; for \; (0 < P < 1)$$

Case.2 $\quad P = 1$

$$\int_1^\infty \frac{dx}{x} = \underset{K\to\infty}{Lt} \int_1^K \frac{dx}{x} = \underset{K\to\infty}{Lt} \ln K = \infty$$

$\therefore \sum u_n$ converges if $P > 1$ and diverges if $0 < P \leq 1$.

18. (b) Test for uniform Convergence of the following series

$$\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} + \frac{\sin 4x}{4\sqrt{4}} + \ldots .\infty.$$

Sol:- Here $u_n(x) = (-1)^{n+1} \dfrac{\sin nx}{n\sqrt{n}}$

$$\therefore |u_n(x)| = \left| \frac{(-1)^{n+1} \sin nx}{n\sqrt{n}} \right| = \frac{\left|(-1)^{n+1} \sin nx\right|}{n\sqrt{n}}$$

$$\leq \frac{1}{n\sqrt{n}} (= M_n) \; \forall \; x \in R$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent, therefore by Weierstrass - M - test, the given series is uniformly convergent for all real *x*.

19. (a) Obtain the Fourier series of the following function with period $T = 2\pi$.

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Sol:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ______(1)

Then $a_o = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\, dx + \int_{0}^{\pi} x\ dx\right] = -\pi/2$

$$a_o = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi \sin nx}{n}\right)_{-\pi}^{0} + \left\{x\frac{\sin nx}{n} + \frac{1}{n^2}\cos nx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\frac{(\cos n\pi - 1)}{n^2} = \frac{1}{n^2\pi}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx\, dx + \int_{0}^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi \cos nx}{n}\right)_{-\pi}^{0} + \left\{\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right\}_0^{\pi}\right]$$

$$= \frac{1}{n}(1-2\cos n\pi) = \frac{1}{n}\left(1-2(-1)^n\right)$$

Substituting the values of a_o, a_n, b_n in (1) we get

$$f(x) = \boxed{\frac{-\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{\left((-1)^n - 1\right)}{n^2\pi}\cos nx + \frac{1}{n}\left(1-2(-1)^n\right)\sin nx\right]}$$

19. (b) Express $f(x) = |x|, -\pi < x < \pi$ as Fourier series.

Hence obtain $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \dots \infty$.

Sol:- Since $f(-x) = |-x| = |x| = f(x)$

$\therefore f(x)$ is an even function and hence $b_n = 0$.

Let $f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$

Then $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right] = 0 \text{ if } n \text{ is even}$$

$$= \frac{-4}{n^2\pi} \text{ if } n \text{ is odd}$$

$$\therefore f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots..\right)$$

Putting $x = 0$ in the above result, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots..\right)$$

$$\Rightarrow \boxed{1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots. = \frac{\pi^2}{8}}.$$

20. (a) Obtain the half range Cosine series for the function

$f(x)=(x+1)^2,\ 0<x<1.$

Sol:- $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}.$

Here $f(x)=(x+1)^2,\ l=1.$

$\therefore (x+1)^2=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x.$

$$a_0=\frac{2}{l}\int_0^l f(x)d(x)$$

$$=2\int_0^1 (x+1)^2\,dx$$

$$=2\int_0^1 (x^2+1+2x)dx=2\left(\frac{x^3}{3}+x+x^2\right)_0^1$$

$$=2\left(\frac{1}{3}+1+1\right)=\frac{14}{3}.$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$=2\int_0^1 (x+1)^2\cos(n\pi x)dx$$

$$=2\left[(x+1)^2\frac{\sin n\pi x}{n\pi}-2(x+1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right.$$

$$+ 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\Bigg]_0^1$$

$$= 2\left[4\frac{\sin n\pi}{n\pi}+\frac{4\cos n\pi}{n^2\pi^2}-\frac{2\sin n\pi}{n^3\pi^3}-\frac{2}{n^2\pi^2}\right]$$

$$= \frac{2}{n^2\pi^2}(4\cos n\pi - 2)$$

$$= \frac{4}{n^2\pi^2}(2\cos n\pi - 1)$$

$$= \frac{4}{n^2\pi^2}\left(2(-1)^n - 1\right)$$

$$\therefore f(x) = \frac{7}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(2(-1)^n - 1\right)\cos n\pi x.$$

20. (b) Obtain the Fourier series of *y* for the given values of *x* neglecting the harmonics above the second.

x	0	1	2	3	4	5
y	9	18	24	28	26	20.

Sol:- Let the Fourier series to represent *y* be

$$y = \frac{a_0}{2}+\left(a_1\cos\frac{\pi x}{3}+b_1\sin\frac{\pi x}{3}\right)+$$

$$\left(a_2\cos\frac{2\pi x}{3}+b_2\sin\frac{2\pi x}{3}\right)+\ldots\ldots$$

The values of $x, y, \sin\frac{\pi x}{3}$, $\cos\frac{\pi x}{3}$ are tabulated below.

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$\cos\frac{2\pi x}{3}$	$\sin\frac{2\pi x}{3}$
0	9	1	0	1	0
1	18	$1/2$	$\sqrt{3}/2$	$-1/2$	$\sqrt{3}/2$
2	24	$-1/2$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/2$
3	28	-1	0	1	0
4	26	$-1/2$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}/2$
5	20	$1/2$	$-\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/2$

Using the values in the above table, we have

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}(125) = 41.67$$

$$a_1 = \frac{2}{n}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}$$

$$\left[9 + 1/2(18) - \frac{1}{2}(24) - (28) - \frac{1}{2}(26) + \frac{1}{2}(20)\right]$$

$$= \frac{1}{3}(25) - 8.33.$$

$$b_1 = \frac{2}{n}\sum y\sin\frac{\pi x}{3}.$$

$$= \frac{2}{6}\left[0 + 18\left(\frac{\sqrt{3}}{2}\right) + 24\left(\frac{\sqrt{3}}{2}\right) + 0 + 26\left(\frac{-\sqrt{3}}{2}\right) + 20\left(\frac{-\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{3}\left[\frac{\sqrt{3}}{2}(18+24-26-20)\right]$$

$$= \frac{\sqrt{3}}{6}(-4) = \frac{-2\sqrt{3}}{3} = -1.15.$$

$$a_2 = \frac{2}{n}\sum y\cos\left(\frac{2\pi x}{3}\right)$$

$$= \frac{2}{6}\left[9+18\left(-\tfrac{1}{2}\right)+24\left(-\tfrac{1}{2}\right)+28(1)+26\left(-\tfrac{1}{2}\right)+20\left(-\tfrac{1}{2}\right)\right]$$

$$= \frac{1}{3}\left[\cancel{9}-\cancel{9}-12+28-13-10\right]$$

$$= \frac{-7}{3} = -2.33$$

$$b_2 = \frac{2}{n}\sum y\sin\left(\frac{2\pi x}{3}\right)$$

$$= \frac{2}{6}\left[9(0)+18\left(\frac{\sqrt{3}}{2}\right)+24\left(\frac{-\sqrt{3}}{2}\right)\right.$$

$$\left.+\ 28(0)+26\left(\frac{\sqrt{3}}{2}\right)+20\left(\frac{-\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{3}\left[9\sqrt{3}-12\sqrt{3}+13\sqrt{3}-10\sqrt{3}\right]$$

$$= \frac{\sqrt{3}}{3}(0) = 0.$$

$$\therefore\ y = 20.835 - 8.33\cos\frac{\pi x}{3} - 1.15\sin\left(\frac{\pi x}{3}\right)$$

$$-2.33\cos\left(\frac{2\pi x}{3}\right) + (0)\sin\left(\frac{2\pi x}{3}\right) + ...$$

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B.E / B.Tech. Ist Year

MATHEMATICS - II

(Common for all branches)

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1. a) Find rank of $\begin{bmatrix} 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 \\ 15 & 16 & 17 & 18 \end{bmatrix}$

Sol : - Given that A = $\begin{bmatrix} 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 \\ 15 & 16 & 17 & 18 \end{bmatrix}$

Replace $R_1 \to -(R_1 - R_2)$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 \\ 15 & 16 & 17 & 18 \end{bmatrix}$$

Replace $R_2 \to R_2 - 5R,$

$R_3 \to R_3 - 10\,R,$

$R_4 \to R_4 - 15\,R,$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Replace $R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A is 2 since no.of non-zero rows is 2.

1. b) Find A^5 if $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ Using Cayley - Hamilton Theorem.

Sol:- The characteristic equation of A is given by

$$1 A - \lambda I 1 = 0$$

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 12 - 7\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$$

Using cayley - Hamilton theorem

$$A^2 - 7A + 10I = 0$$

$$A^2 = 7A - 10\,I$$

$$= 7 \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 7 \\ 14 & 21 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$$

$$A^4 = (7A - 10I)(7A - 10I)$$

$$= 49A^2 - 140A + 100I$$

$$= 49(7A - 10I) - 140A - 100I$$

$$A^5 = 203A^2 - 390A$$

$$= 203(7A - 10I) - 390A$$

$$= 1421A - 2030I - 390A$$

$$= 1031A - 2030I$$

$$= 1031 \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 2030 & 0 \\ 0 & 2030 \end{bmatrix}$$

$$\boxed{A^5 = \begin{bmatrix} 2094 & 1031 \\ 2062 & 1063 \end{bmatrix}}$$

2. (a) Using elementary row operation find inverse of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

Sol : - Consider A/I = and apply elementary row operations on both A and I unil A gets transformed to I.

$$A/I = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 1 & 2 & 3 & \vdots & 0 & 1 & 0 \\ 1 & 4 & 9 & \vdots & 0 & 1 & 1 \end{bmatrix}$$

Replace $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 2 & : & -1 & 1 & 0 \\ 0 & 3 & 8 & : & -1 & 1 & 1 \end{bmatrix}$$

Replace $R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 2 & : & -1 & 1 & 0 \\ 0 & 0 & 2 & : & 2 & -2 & 1 \end{bmatrix}$$

Replace $R_3 \to R_3 / 2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 2 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & 1 & -1 & 1/2 \end{bmatrix}$$

Replace $R_1 \rightarrow R_1 - R_3$

$R_2 \rightarrow R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 & : & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & : & -3 & 3 & -1 \\ 0 & 0 & 1 & : & 1 & -1 & \frac{1}{2} \end{bmatrix}$$

Replace $R_1 \rightarrow R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 & -2 & 1/2 \\ 0 & 1 & 0 & : & -3 & 3 & -1 \\ 0 & 0 & 1 & : & 1 & -1 & 1/2 \end{bmatrix} = [I / A^{-1}]$$

Thus $\boxed{A^{-1} = \begin{bmatrix} 3 & -2 & 1/2 \\ -3 & 3 & -1 \\ 1 & -1 & 1/2 \end{bmatrix}}$

2. (b) Using sylvester's theorem, find $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ **find** A^{100}.

Sol :- $f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 1 \end{vmatrix}$

$$f(\lambda) = (\lambda - 2)(\lambda - 1) = 0 \ or \ \lambda_1 = 1, \ \lambda_2 = 2$$

$$f(\lambda) = \lambda^2 - 3\lambda + 2, \ f'(\lambda) = 2\lambda - 3 \ or \ \lambda_1 = 1, \lambda_2 = 2$$

$$f'(2) = 4 - 3 = 1, \ f'(1) = 2 - 3 = -1$$

$[f(\lambda)] =$ Adjoint of the matrix of the matrix $[\lambda I - A]$

$$Z(\lambda_1) = Z(1) = \frac{[f(1)]}{f'(1)} = \frac{1}{-1}\begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$$

$$Z(\lambda_2) = Z(2) = \frac{[f(2)]}{f'(2)} = \frac{1}{1}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

By Sylvester theorem

$$P(A) = P(\lambda_1).\, Z(\lambda_1) + P(\lambda_2).\, Z(\lambda_2)$$

$$A^{100} = P(\lambda_1)\, Z(\lambda_1) + P(\lambda_2)\, Z(\lambda_2)$$

$$= \lambda_1^{100}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda_2^{100}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 1^{100}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 2^{100}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2^{100} & 0 \\ 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2^{100} & 0 \\ 0 & 1 \end{bmatrix}}$$

3) a) Using Gauss Elimination Method Solve

$$3x + 4y + 5z = 40,\ 2x - 3y + 4z = 13,\quad x + y + z = 9$$

Sol:- Consider the augumanted matrix $[A/B]$

$$[A/B] = \left[\begin{array}{ccc|c} 3 & 4 & 5 & 40 \\ 2 & -3 & 4 & 13 \\ 1 & 1 & 1 & 9 \end{array}\right]$$

Interchanging $R_1 \Leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

Replace $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

Interchanging $R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & -5 & 2 & -5 \end{array}\right]$$

Replace $R_3 \to R_3 + 5R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

Interchanging $R_3 \to R_3 / 12$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

By back substitution:

$$z = 5$$

$$y + 2z = 13$$

$$y = 3$$

$$x + y + z = 9$$

$$x = 9 - 3 - 5$$
$$= 1.$$

Thus $\boxed{x = 1,\ y = 3,\ z = 5}$

3) b) Apply Crout's for LU factorisation mthod to solve the eqations $3x + 2y + 7z = 4; 2x + 3y + z = 5;$ $3x + 4y + z = 7.$

Sol:- Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} (i.e. A),$

So that

i) $u_{11} = 3, u_{12} = 2, \qquad u_{13} = 7.$

ii) $l_{21}u_{11} = 12, \qquad \therefore l_{21} = 2/3$

$l_{31}u_{11} = 3, \qquad \therefore l_{31} = 1.$

iii) $l_{21}u_{12} + u_{22} = 3, \qquad \therefore u_{22} = 5/3,$

$l_{21}u_{13} + u_{23} = 1, \qquad \therefore u_{23} = -11/3.$

iv) $l_{31}u_{12} + l_{32}u_{22} = 4, \qquad \therefore l_{32} = 6/5.$

v) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$

$\therefore u_{33} = -8/5.$

Thus $$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

Writing UX=V, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Solving this system, we have $\upsilon_1 = 4,$

$$\frac{2}{3}\upsilon_1 + \upsilon_2 = 5 \qquad \text{or} \qquad \upsilon_2 = \frac{7}{3}$$

$$\upsilon_1 + \frac{6}{5}\upsilon_2 + \upsilon_3 = 7 \qquad \text{or} \qquad \upsilon_3 = \frac{1}{5}$$

Hence the original system becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

i.e $3x + 2y + 7z = 4$; $\frac{5}{3}y - \frac{11}{3}z = \frac{7}{3}; -\frac{8}{5}z = \frac{1}{5}$

By back-substitution, we have $z = -1/8$, $y = 9/8$ *and* $\boxed{x = 7/8.}$

4) a) **Solve** $e^y\left(1 + \frac{dy}{dx}\right) = e^x$

Sol : **Given,**

$$1 + \frac{dy}{dx} = \frac{e^x}{e^y}$$

$$1 + \frac{dy}{dx} = e^{x-y} \qquad \text{-----------(1)}$$

Let $x - y = z$

$$1 - \frac{dy}{dx} = \frac{dz}{dx}$$

Substituting in ------------(2)

$$2 - \frac{dz}{dx} = e^z$$

$$\frac{dz}{dx} = 2 - e^z$$

$$\frac{dz}{2 - e^z} = dx$$

$$\int \frac{dz}{2 - e^z} = \int dx$$

Integrating,

$$\frac{z}{2} - \frac{1}{2} \log(2 - e^z) = x + c$$

Substutute value of z

$$\frac{x-y}{2} - \frac{1}{2} \log(2 - e^{x-y}) = x + c$$

$$x - y - \log(2 - e^{x-y}) = 2x + c$$

$$\boxed{\therefore\ x + y + \log(2 - e^{x-y}) = c}$$

4. (b) Use the Gram Schmidt formula to obtain an orthrnormal set of given Linearly independent set (6,0), (2,1)

Sol :- The given set of vector is (6,0), (2,1)

The new set of orthonormal vectors are given by α_1, α_2

Where $\alpha_1 = \frac{(6,0)}{\sqrt{36}} = (1,0)$

$$\alpha_2 = (2,1) - \frac{[(2,1).(1,0)]}{1}(1,0)$$

$$= (2,1) - 2(1,0) = (0,1)$$

$\therefore$ orthonormal set : $\boxed{(1,0),\ (0,1)}$

5. a) **Solve :** $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$

Sol :- The given equation is $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$

Here $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Put $x = X + h$

$y = Y + k$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Eq (1) becomes

$$\frac{dY}{dX} = \frac{(Y+X) + (k+h-2)}{(Y-X)\ (k-h-4)}$$

$$k+h-2=0$$

$$k-h-4=0 \Rightarrow h=-1 \ and \ k=3$$

Eq (1) become reduced to

$$\frac{dY}{dX} = \frac{Y+X}{Y-X}$$

which is homogeneous equation

put $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dX} = \frac{vX+X}{vX-X} = \frac{v+1}{v-1}$$

$$v\frac{dX}{X} = \frac{v+1}{v-1} - v = \frac{1+2v-v^2}{v-1}$$

$$\frac{dX}{X} = \frac{v-1}{-(v^2-2v-1)}\, dv$$

$$2\frac{dX}{X} + \frac{2v-2}{v^2-2v-1}\, dv = 0$$

$$2\log X + \log(v^2-2v-1) = \log e$$

$$X^2\left(\frac{Y^2}{X^2} - \frac{2Y}{X} - 1\right) = C$$

$$Y^2 - 2XY - X^2 = c$$

But $x = X + h = X - 1$ *i.e* $X = x + 1$

Similarly $Y = y - 3$

$\therefore$ The general solution is

$$\boxed{(y-3)^2 - 2(x+1)(y-3) - (x+1)^2 = C}$$

5. b) Solve : $x^2 y^2 (ydx + 2xdy) + xy(ydx - xdy) = 0$

Sol :- $x^2 y^3 dx + 2x^3 y^2 dy + xy^2 dx - x^2 ydy = 0$

Rearranging the terms

$$x^2 y^2 (ydx + 2xdy) + xy(ydx - xdy) = 0$$

Comparing this with

$$x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = c$$

here $a = 2,\ b = 2,\ m = 1,\ n = 2$

$c = 1,\ d = 1,\ p = 1,\ q = -1$

Also $mp - nq = 1 + 2 = 3 \neq 0$

The unknown constants are determined from the following

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}$$

i.e. $\dfrac{2+h+1}{1} = \dfrac{b+k+1}{2}$

$$2h - k + 3 = 0$$

$$\frac{C+h+1}{p} = \frac{d+k+1}{q}$$

i.e $\dfrac{1+h+1}{1} = \dfrac{1+k+1}{-1}$

$$h + k + 4 = 0$$

solging we get $h = -\frac{7}{3}, \ k = -\frac{5}{3}$

The required integrating factor

$$x^{-7/3} \ . \ y^{-5/3}$$

Multiplying D.E by integrating factor

$$x^{-7/3}. \ y^{-5/3} x^2 y^2 (ydx + 2xdy) + x^{-7/3}. \ y^{-5/3} xy \ (ydx - xdy) = 0$$

The equation is now exact

$$\therefore \int_{y=constnt} Mdx + \int_{Terms\ not\ containing\ x} Ndy = 0$$

$$\int x^{-7/3}. \ y^{-5/3} x^2 y^2 ydx + \int x^{-7/3}. \ y^{-5/3} xy \ ydx = c$$

$$\boxed{\frac{3}{2} x^{2/3} y^{5/3} - 3x^{-1/3} y^{1/3} = C}$$

6. a) Find orthogonal trajectories of family of cuares

$$r^n \ \mathbf{sin} \ n\theta = a^n$$

Sol:- $r^n \sin n\theta = a^n$

$$\Rightarrow n \log r + \log \sin n\theta = n \log a$$

Differentiating w.r.t θ

$$\frac{n}{r} . \frac{dr}{d\theta} + \frac{\cos n\theta}{Sin\, n\theta} n = 0$$

$$\Rightarrow \frac{dr}{d\theta} + r \cot n\theta = 0$$

$$\frac{dr}{d\theta} = -r\ Cot\ n\theta$$

Replacing $\frac{dr}{d\theta}$ $by - r^2 \frac{d\theta}{dr}$ we get the trajectories of given of curves.

$$\Rightarrow -r^2 \frac{d\theta}{dr} = -r \cot n\theta$$

$$\frac{dr}{r} = \frac{d\theta}{\cot n\theta}$$

$$\Rightarrow \frac{dr}{r} = \tan n\theta\ d\theta$$

Integrating, $\log r = \frac{-\log(\cos n\theta)}{n} + \log c$

$$\Rightarrow \log r = \frac{\log \sec n\theta}{n} + \log c$$

$$\Rightarrow n \log r = \log \sec n\theta + n \log c$$

$$\Rightarrow n \log r = \log \sec n\theta + n \log c$$

$$r^n = c^n \sec n\theta$$

which is required equation of trajectories.

7. a) Solve : $\frac{d^3y}{dx^3} + y = 5 + Sin2x$

Sol : $\frac{d^3y}{dx^3} + y = 5 + Sin2x$

The operation for M of D.E is

$$(D^3 + 1)y = \sin 2x$$

The auxilary equation is

$$m^3 + 1 = 0$$

$$(m+1)\ (m^2 - m+1) = 0$$

$$m = -1,\ \frac{1+\sqrt{3}\ i}{2},\ \frac{1-\sqrt{3}\ i}{2}$$

Hence the CF

$$CF = C_1 e^{-x} + e^{X/2}\left[c_2 \cos \frac{\sqrt{3}}{2} X + C_3\, Sin\, \frac{\sqrt{3}}{2} X\right]$$

Particular Integral

$$PI = \frac{1}{D^3+1}(5 + \sin 2x)$$

$$= \frac{5}{D^3+1} + \frac{\sin 2x}{D^3+1}$$

$$= \frac{5}{D^3+1} + \frac{\sin 2x}{D^2.\ D+1}$$

$$= \frac{5}{0+1} + \frac{\sin 2x}{-2^2.\ D+1}$$

$$= 5 + \frac{\sin 2x}{-4D+1}$$

$$= 5 - \frac{[4D+1]\ Sin\, 2x}{16D^2 - 1}$$

$$= 5 - \frac{[4D+1]}{16(-2^2)-1}\ Sin\, 2x$$

$$= 5 - \frac{[4D+1]}{-65}\ Sin\, 2x$$

$$= 5 + \frac{1}{65}\ (4D+1) Sin\, 2x$$

$$= 5 + \frac{1}{65}\ [4\ .\ 2Cos\, 2x + 2\sin 2x]$$

$$= 5 + \frac{8}{65}\cos 2x + \frac{2}{65}\sin 2x$$

The complete genral eqation is

$$y = C.F + P.I$$

$$= c_1 e^{-x} + e^{x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}X\right]$$

$$\boxed{5 + \frac{+2}{65}\sin 2x + \frac{8}{65}\cos 2x}$$

7) b) Solve : $\frac{dx}{dt} + 2y = e^t, \frac{dy}{dt} - 2x = e^{-t}$

Sol: $\frac{dx}{dt} + 2y = e^t$ -------(1)

$\frac{dy}{dt} - 2x = e^t$ -------(2)

Solving (2) for x we get

$2x = \frac{dy}{dt} - e^{-t}$ -------(3)

Differentiating (3) w.r.t (t)

$2\frac{dx}{dt} = \frac{d^2y}{dt^2} + e^{-t}$ -------(4)

Substitute (1) in ------(4)

$$2(e^t - 2y) = \frac{d^2y}{dt^2} + e^{-t}$$

$$\Rightarrow \frac{d^2y}{dt^2} + e^{-t} = 2(e^t - 2y)$$

$$\frac{d^2y}{dt^2} + 4y = 2e^t - e^{-t}$$

This is a Second order differential equation

$\therefore$ C.F $C_1 \cos t + C_2 \sin t$

$$\text{P.I } \frac{1}{D^2+4}\left(2e^t - e^{-t}\right)$$

$$= \frac{2e^t}{5} - \frac{e^{-t}}{5}$$

$$y\left(t\right) = C_1 \cos\, t + C_2 \sin t + \frac{1}{5}\left(2e^t - e^{-t}\right) \quad \text{------(5)}$$

substituting (5) in (3)

$$x(t) = \frac{1}{2}\left(-C_1 \sin t + C_2 \cos t + \frac{1}{5}\left(2e^t + e^{-t}\right) - e^{-t}\right) \text{------(6)}$$

The general equations of (1) & (2) is given by (5) &(6)

$$\boxed{i.e\ \frac{1}{2}\left(-C_1 \sin t + C_2 \cos t\right) + \frac{2e^t}{5} - \frac{4}{5}e^{-t}}$$

8) a) **Solve :** $(2x+3)^2 \dfrac{d^2y}{dx^2} - (2x+3)\dfrac{dy}{dx} - 12y = 6x$

Sol : Here $a_0 = 1,\ b_0 = -1,\ c_0 = -12$

$$Q(x) = 6x,\ a = 2,\ b = 3$$

Replace $2x+3 = e^t$. The given equation reduces to linear equation by using equations in Legendre D.E.

The legendre D.E equations are

$$x = \frac{e^t - b}{a}$$

$$xDy = aDy$$

$$x^2D^2y = a^2[D^2 - D]$$

$\therefore$ The equation becomes

$$4\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right] - 2\frac{dy}{dt} - 12y = 3(e^t - 3)$$

$$4\frac{d^2y}{dt} - \frac{6dy}{dt} - 12y = 3(e^t - 3)$$

$$2\frac{d^2y}{dt} - \frac{3dy}{dt} - 6y = \frac{1}{2}(3e^t - 9)$$

C.F : The A.E. is $2m^2 - 3m - 6 = 0$ with roots $m = \dfrac{3 \pm \sqrt{57}}{4}$

so that C.F. y_c is

$$y_c = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)t}$$

P.I: $y_p = \dfrac{1}{2D^2 - 3D - 6} \cdot \dfrac{1}{2}(3e^t - 9)$

$$= \frac{3}{2}\frac{e^t}{2D^2 - 3D - 6} - \frac{9}{2}\frac{1}{2D^2 - 3D - 6}$$

$$= \frac{3}{2}\frac{1}{2-3-6}e^t - \frac{9}{2} \cdot \frac{1}{0-0-6}$$

$$= -\frac{3}{14}e^t + \frac{3}{4}$$

Then G.S: $y = y_c + y_p$

$$y = c_1\, e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2\, e^{\left(\frac{3-\sqrt{57}}{4}\right)t} - \frac{3}{14}e^t + \frac{3}{4}$$

Replacing t by in $(2x+3)$, we get

$$\boxed{y(x) = c_1(2x+3)^{\frac{3+\sqrt{57}}{4}} + c_2(2x+3)^{\frac{3-\sqrt{57}}{4}} - \frac{3}{14}(2x+3) + \frac{3}{4}}.$$

8. b) **solve:** $\frac{d^2y}{dx^2} + y = \tan x$

Sol :- $\frac{d^2y}{dx^2} + y = \tan x$ ---------(1)

Consider, $\frac{d^2y}{dx^2} + y = 0$ ---------(2)

$$(D^2+1)\,y = 0$$

$y = c_1 \cos x + c_2 \sin x,$ suppose

Let the P.I is

$y = A\cos x + B\sin x$ ---------(3)

Where A & B are variables.

Differentiating (3) w.r.t x

$$\frac{dy}{dx} = -A\sin x + A^1\cos x + B\cos x + B^1\sin x$$

Choose A and B such that

$$A^1 \cos x + B^1 \sin x = 0 \qquad \text{------(4)}$$

$$\frac{dy}{dx} = -A\sin x + B\cos x \qquad \text{------(5)}$$

Differentiating (5) w.r.t x

$$\frac{d^2y}{dx^2} = -A\cos x - A^1 \sin x - B\sin x + B^1 \cos x \qquad \text{----(6)}$$

Subsituting (3), (6) in (1)

$$\therefore\ -A^1 \sin x + B^1 \cos x - \tan x = 0$$

Solving (4) & (7)

$$A^1 = -\sin^2 x.\ \sec x = -\left(\frac{-\cos^2 x + 1}{\cos x}\right)$$

$$= -\sec x + \cos x$$

$$B^1 = \sin x$$

Integrating we get

$$A = -\log(\sec x + \tan x) + \sin x.$$

$$B = -\cos x + a_2$$

Complete solution is $C_1 \cos x + C_2 \sin x$

$$-\cos x \log\left(\sec x + \tan x\right)$$

Method - II

From (3) $y_1 = \cos x \qquad y_2 = \sin x$

$$y_1^1 = -\sin x \qquad y_2^1 = \cos x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$P.I = -y_1 \int \frac{y_2 R}{n1} dx + y_2 \int \frac{y_1 R}{n1} dx \text{ whose } R = \sec x$$

$$R = \tan x$$

$$P.I = -\cos x \int \sin x . \tan x \, dx + \sin x \int \cos x . \tan x \, dx$$

Complete Solution = $c_1 \cos x + c_2 \sin x - \log(\sec x + \tan x)$

9. a) Find Laplace transfor of

i) $e^t . \cos^2 t$ **ii)** $\sin^3(4t)$ **ii)** $e^t . \cos^2 t$

Sol :- $L[e^t \cos^2 t]$

$$L[\cos^2 t] = L\left[\frac{1+\cos 2\theta}{2}\right]$$

$$= L\left[\frac{1}{2}\right] + \frac{1}{2} L[\cos 2\theta]$$

$$= \frac{1}{2S} + \frac{1}{2} \frac{S}{S^2+4}$$

By shifting theorem, replace S by S-1

$$\therefore\ L[e^t \cos^2 t] = \frac{1}{2(S-1)} + \frac{1}{2}\ \frac{S-1}{(S-1)^2+4}$$

$$= \frac{1}{2}\left[\frac{1}{S-1} + \frac{S-1}{S^2-2S+5}\right]$$

$$= \frac{1}{2}\left[\frac{S^2-2S+5+S^2-2S+1}{(S-1)\ (S^2-2S+5)}\right]$$

$$= \frac{1}{2}\left[\frac{2S^2-4S+6}{(S-1)\ (S^2-2S+5)}\right]$$

$$= \frac{2S^2-4S+3}{(S-1)\ (S^2-2S+5)}$$

ii) $Sin^3(4t)$

Sol : $L[Sin^3(4t)]$

$$L[Sin^3(4t)] = L\left[\frac{3Sin(4t) - Sin3(4t)}{4}\right]$$

$$= L\left[\frac{3Sin(4t)}{4}\right] - L\left[\frac{Sin3(4t)}{4}\right]$$

$$= \frac{3}{4}\ \frac{4}{S^2+4^2} - \frac{1}{4}\ \frac{12}{S^2+144}$$

$$= \frac{12}{4}\left[\frac{1}{S^2+16} - \frac{1}{S^2+144}\right]$$

$$= 3\left[\frac{\cancel{S^2}+144-\cancel{S^2}-16}{\left(S^2+16\right)\left(S^2+144\right)}\right] = \frac{384}{(S^2+16)\ (S^2+144)}$$

9. b) Find inverse Laplace transform of

i) $\dfrac{S-2}{(S^2-4S+2)^2}$ **ii)** $\dfrac{S^3}{S^4-16}$

Sol:- i) $\not{S}\ L^{-1}\left[\dfrac{S-2}{(S^2-4s+2)^2}\right]$

$$= L^{-1}\ \frac{S-2}{[(S-2)^2-2]^2}$$

$$= e^{2S}L^{-1}\ \frac{S}{(S^2-2)^2}$$

$$= e^{2S},-\frac{1}{2}\ L^{-1}\left[\frac{d}{ds}\left(\frac{1}{S^2-1}\right)\right]=-\frac{1}{2}e^{2t}t.L^{-1}\left(\frac{1}{S^2-2}\right)$$

$$=\frac{1}{2}e^{2t}\,t\,\frac{\sin h\sqrt{2}\,t}{\sqrt{2}}$$

ii) $L^{-1}\left[\dfrac{S^3}{S^4-16}\right]$

$$L^{-1}\left[\frac{S^3}{S^4-2^4}\right]$$

Sol :- $S^4-2^4=(S^2)^2-(2^2)^2$

$$=(S^2+4)\ (S^2-4)$$

Let $\dfrac{S^3}{S^4 - 2^4} = \dfrac{AS+B}{S^2+4} + \dfrac{CS+D}{S^2-4}$

$$S^3 = (AS+B)\,(S^2-4) + (CS+D)\,(S^2+4)$$

By solving we get

$$A = \tfrac{1}{2},\ B = 0,\ C = \tfrac{1}{2},\ 0 = 0$$

$$\frac{S^3}{S^4 - 2^4} = \frac{1S}{2(S^2+4)} + \frac{1}{2(S^2-4)}$$

$$L^{-1}\left[\frac{S^3}{S^4-2^4}\right] = \frac{1}{2}\,L^{-1}\left[\frac{S}{S^2+4}\right] + \frac{1}{2}\,L^{-1}\left[\frac{1}{S^2-4}\right]$$

$$= \frac{1}{2}\cos 2t + \frac{1}{2}\,L^{-1}\left[\frac{1}{4(S-2)} - \frac{1}{4(S+2)}\right]$$

$$= \boxed{\frac{1}{2}\cos 2t + \frac{1}{8}\,e^{2t} - \frac{1}{8}\,e^{-2t}}$$

10) a) Using convolution theorem, find inverse transform of $\dfrac{S^2}{(S^2+4)^2}$

Sol :- Rewriting

$$\frac{S^2}{(S^2+4)^2} = \frac{S}{(S^2+2^2)} \cdot \frac{S}{(S^2+2^2)}$$

$$L^{-1}\left\{\frac{S}{(S^2+a^2)}\right\} = \cos at$$

$$\therefore \; L^{-1}\,\frac{S}{(S^2+2^2)} = \cos 2t$$

By convolution theorem,

$$L^{-1}\left[\frac{S^2}{(S^2+2^2)\,(S^2+2^2)}\right] = L^{-1}\left[\frac{S}{S^2+2^2}\cdot\frac{S}{S^2+2^2}\right]$$

$$= \int_0^t \cos 2u \,.\, \cos 2(t-u)du$$

$$= \frac{1}{2}\int_0^t \cos(2u+2t-2u) + \cos(2u-2t+2u]du$$

$$= \frac{1}{2}\int_0^t [\cos 2t + \cos 2(2u-t)]\,du$$

$$= \left[\frac{1}{2}\,u.\cos 2t + \frac{\sin 2(2u-t)}{4}\right]_{u=0}^{t}$$

$$= \left[\frac{1}{2}\,t.\cos 2t + \frac{\sin 2t}{4} - \frac{\sin(-2t)}{4}\right]$$

$$= \left[\frac{1}{2}\,t.\cos 2t + \frac{\sin 2t}{2}\right]$$

10. b) Solve using Laplace transform

$$y^{11} + 4y = e^t, \; y^1(0) = 0, \; y(0) = 1$$

Sol : Given $y^{11} + 4y = e^t$

Applying Laplace transform, we have

$$L[y^{11} + 4y] = L[e^t]$$

$$[S^2 y(S) - Sy(0) - y^1(0)] + 4y = \frac{-1}{S-1}$$

$$y^1(0) = 0, \quad y(0) = 1$$

$$\left[S^2\, y(S) - S\right] + 4y = \frac{1}{S-1}$$

$$y(S) = \frac{1}{(S-1)(S^2+4)} + \frac{S}{(S^2+4)}$$

Taking inverse laplace transform

$$y(t) = L^{-1}\left[\frac{1}{(S-1)\,(S^2+4)}\right] + L^{-1}\left[\frac{S}{(S^2+4)}\right]$$

Final $L^{-1}\left[\frac{1}{(S-1)\,(S^2+4)}\right], \; L^{-1}\left[\frac{S}{(S^2+4)}\right]$

$$\frac{1}{(S-1)\,(S^2+4)} = \frac{A}{(S-1)} + \frac{BS+C}{(S^2+4)}$$

$$= \frac{1}{5(S-1)} - \frac{1}{5}\,\frac{(S+1)}{(S^2+4)}$$

$$L^{-1}\left[\frac{1}{(S-1)(S^2+4)}\right] = L^{-1}\left[\frac{1}{5(S-1)} - \frac{1}{5}\,\frac{(S+1)}{(S^2+4)}\right]$$

$$= \frac{1}{5}\,L^{-1}\left[\frac{1}{S-1}\right] - \frac{1}{5}\,L^{-1}\left[\frac{S+1}{S^2+4}\right]$$

$$= \frac{1}{5}\,L^{-1}\left[\frac{1}{S-1}\right] - \frac{1}{5}\,L^{-1}\left[\frac{S}{S^2+4}\right] - \frac{1}{5}\,L^{-1}\left[\frac{1}{S^2+4}\right]$$

$$= \frac{1}{5}\,e^t - \frac{1}{5}\cos 2t - \frac{1}{5}\,\frac{1}{2}\sin 2t$$

$$L^{-1}\left[\frac{S}{S^2+4}\right] = \cos 2t$$

$$\therefore\ y(t) = \frac{1}{5}e^t - \frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t + \cos 2t$$

$$= \frac{1}{5}e^t - \frac{4}{5}\cos 2t - \frac{1}{5\times 2}\sin 2t$$

$$= \boxed{\frac{1}{5}\left(e^t + 4\cos 2t - \frac{\sin 2t}{2}\right)}$$

11. a) Determine the rank of the following matrix by reducing it to a canonical matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Sol :- Canonical form is also called diagonal form

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_1$$
$$R_4 \to 3R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -8 & -10 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$R_3 \to 5R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -8 & -10 \\ 0 & 0 & 24 & 50 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$C_2 \to C_2 + C_1$$
$$C_3 \to C_3 - 2C_1$$
$$C_4 \to C_4 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & -10 \\ 0 & 0 & 24 & 50 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$C_3 \to 5C_3 + 8C_2$$
$$C_4 \to C_4 + 2C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 120 & 100 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$C_4 \to 6C_4 - 5C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 120 & 0 \\ 0 & 0 & 0 & 60 \end{bmatrix} = B$$

$$|B| \neq 0$$

$\therefore$ **The rank of the given matrix is 4.**

11. b) Using caylay - Hamilton theorem, find the

Inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

Sol :- Given that $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = A$

The characteric matrix is $\left[A-\lambda I\right]$

$$= \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 2-\lambda & 3 \\ 1 & 4 & 9-\lambda \end{bmatrix}$$

Charactersic equation is $\left|A-\lambda I\right| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 2-\lambda & 3 \\ 1 & 4 & 9-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(9-\lambda)-12\right]-$$

$$1\left[(9-\lambda)-3\right]+1\left[4-(2-\lambda)\right]=0$$

$$(1-\lambda)\left[\left(18-2\lambda-9\lambda+\lambda^2-12\right)\right]$$

$$-1[9-\lambda-3]+1\,[4-2+\lambda]=0$$

$$(1-\lambda)\left[\left(\lambda^2-11\lambda+6\right)\right]-1\,[-\lambda+6]+1\,[\lambda+2]=0$$

$$\lambda^2-11\lambda+\not{6}-\lambda^3+11\lambda^2-6\lambda+\lambda-\not{6}+\lambda+2=0$$

$$-\lambda^3+12\lambda^2-15\lambda+2=0$$

$$\Rightarrow \lambda^3-12\lambda^2+15\lambda-2=0 \qquad \_\_\_\_\_(1)$$

Accroding to the Caylay-Hamiton theorem, put $\lambda = A$

i.e., $A^3-12A^2+15A-2I=0$

Multiplying through out by A^{-1}, we get

$$2A^{-1}=A^2-12A+15$$

$$A=\begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix} \quad \Rightarrow \; A^2=\begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix}\begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix}$$

$$=\begin{bmatrix}3&7&13\\6&17&34\\14&45&94\end{bmatrix}$$

$$2A^{-1}=\begin{bmatrix}3&7&13\\6&17&34\\14&45&94\end{bmatrix}-\begin{bmatrix}12&12&12\\12&24&36\\12&48&108\end{bmatrix}+\begin{bmatrix}15&0&0\\0&15&0\\0&0&15\end{bmatrix}$$

$$2A^{-1} = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}}$$

12. a) Reduce the following quadratic form to sum of squares by linear tranformation.

Sol :- Given quadratic form is $x^2 + 4y^2 + z^2 + 4xy + 6yz + 2zx$.

The Matrix is $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The diagonal elements are $a_{11} = 1,\ a_{22} = 4,\ a_{33} = 1$

Non-diagonal elements are

$$\therefore\ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 1 \end{bmatrix} \qquad \begin{aligned} a_{12} &= a_{21} = \tfrac{1}{2}(4) = 2 \\ a_{13} &= a_{31} = \tfrac{1}{2}(2) = 1 \\ a_{23} &= a_{32} = \tfrac{1}{2}(6) = 3 \end{aligned}$$

Reduce the given matrix A to the diagonal form by applying elementary transformations.

Row transformations are to be applied on the pre identity matrix and the corresponding column transformations on the post identity matrix.

$$[A]_{3\times 3} = [I]_{3\times 3} \; A[I]_{3\times 3}.$$

Row

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Column

$R_2 - 2R_1,\; R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1,\; C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 + \frac{1}{2}R_3,\; C_2 + \frac{1}{2}C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{2} & 1 & \frac{1}{2} \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{5}{2} & -1 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$

$R_3 - R_2,\; C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5/2 & 1 & +1/2 \\ 3/2 & -1 & +1/2 \end{bmatrix} A \begin{bmatrix} 1 & -5/2 & 3/2 \\ 0 & 1 & -1 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$X^T AX = Y^T P^T APY = Y^T DY$$

The canomical form is

$$\boxed{y_1^2 + y_2^2 - y_3^2}$$

The transformation which reduces the quadratic form is
$X = PY$

That is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -5/2 & 3/2 \\ 0 & 1 & -1 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

12. b) Show that the diagonal elements of a skew-Hermitian matrix are either zero purely imaginary numbers.

Sol :- Let "A" be a skew - Hermitian matrix so that $A^{\Theta} = -A$. Where Θ is conjugate transpose.

Let the $(mn)^{th}$ element of A be $a_{mn} + ib_{mn}$. The elements of A satisfy the relation.

$$-(a_{mn} + ib_{mn}) = (a_{nm} - ib_{nm})$$

For the diagonal elements take $m = n$

$$-\left(a_{nn}+ib_{nn}\right)=a_{nn}-ib_{nn}$$

$$\Rightarrow \; a_{nn}=0$$

That is, the real part of the diagonal elements is zero.

$\therefore$ The diagonal elements of A are of the form ib_{nn}, where b_{nn} is any real number including zero. Hence the diagonal elements of a skew - Hermitian matrix are either zero (if $b_{nn}=0$) or purely imaginary numbers

13. a) Using elementary row operations find the inverse of

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Sol:- Let $A=IA.$

Applying elementary now operations on both sides

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A.$$

$R_2 \to 2R_2 - R_1$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 3 & 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

$R_3 \to 2R_3 - 3R_1$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & -7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -3 & 0 & 2 \end{bmatrix} A$$

$R_3 \to R_3 + 7R_2$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -10 & 14 & 2 \end{bmatrix} A$$

$R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 2 & 0 & -14 \\ 0 & 1 & 5 \\ 0 & 0 & 36 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 0 \\ -1 & 2 & 0 \\ -10 & 14 & 2 \end{bmatrix} A$$

$R_1 \to R_1/2$, $R_3 \to \frac{R_3}{36}$

$$\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \\ -5/18 & 7/18 & 1/18 \end{bmatrix} A$$

$R_2 \to R_1 + 7R_3$

$R_2 \to R_2 - 5R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/18 & -5/18 & 7/18 \\ 7/18 & 1/18 & -5/18 \\ -5/18 & 7/18 & 1/18 \end{bmatrix} A$$

$$\boxed{A^{-1} = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}}$$

13. b) Using Gauss elimination method solve.

$2x + 3y + z = 9, \quad x + 2y + 3z = 6, \quad 3x + y + 2z = 8.$

Sol :- The Augmented matrix

$$(AB) = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 6 \\ 3 & 1 & 2 & 8 \end{array}\right]$$

$R_2 \longleftrightarrow R_1$

$$(AB) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 3 & 1 & 9 \\ 3 & 1 & 2 & 8 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 3 & 1 & 2 & 8 \end{array}\right]$$

$R_3 \to R_3 - 3R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 0 & -5 & -7 & -10 \end{array}\right]$$

$R_3 \to R_3 - 5R_2$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 18 & 5 \end{array}\right]$$

The given sytem of equations is equivalent to

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 5 \end{bmatrix}$$

$x + 2y + 3z = 6, \quad -y - 5z = -3; \; 18z = 5$

$\Rightarrow \quad z = \dfrac{5}{18}, \; y = 3 - 5z = \dfrac{29}{18}$

$x = 6 - 2y - 3z$

$x = 6 - \dfrac{58 + 15}{18} = \dfrac{35}{18}$

$$\boxed{\therefore \; x = \frac{35}{18} \quad y = \frac{29}{18} \quad z = \frac{5}{18}}$$

14. a) Solve the following equations by Cholesky's method

$$2x-4y+3z=1;\ x-2y+4z=3;\ 3x-y+5z=2$$

Sol:- Arrange the equations in the form $AX = B$ such that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$ in the coefficient matrix A.

$$\begin{vmatrix} 3 & -1 & 5 \\ 1 & -2 & 4 \\ 2 & -4 & 3 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{vmatrix} 3 & -1 & 5 \\ 1 & -2 & 4 \\ 2 & -4 & 3 \end{vmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing the elements bothsides, we get

$$l_{11}=3,\ l_{11}u_{12}=-1 \Rightarrow u_{12}=-\frac{1}{3}$$

$$l_{11}u_{13}=5 \Rightarrow u_{13}=\frac{5}{3},\ l_{21}=1,\ l_{31}=2$$

$$l_{21}u_{12}+l_{22}=-2 \Rightarrow l_{22}=-2+\frac{1}{3}=-\frac{5}{3}$$

$$l_{31}u_{12}+l_{32}=-4 \Rightarrow l_{32}=-4+\frac{2}{3}=-\frac{10}{3}$$

$$l_{21}u_{13}+l_{22}u_{23}=4 \Rightarrow 4_{23}=\left(4-\frac{5}{3}\right)\left(\frac{-3}{5}\right)=\frac{-7}{5}$$

$$l_{31}u_{13} + l_{32}\,u_{23} + l_{33} = 3 \;\Rightarrow\; l_{33} = 3 - \frac{10}{3} - \frac{14}{3} = -5$$

$$\therefore \begin{bmatrix} 3 & -1 & 5 \\ 1 & -2 & 4 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -5/3 & 0 \\ 2 & -10/3 & -5 \end{bmatrix} \begin{bmatrix} 1 & -1/3 & 5/3 \\ 0 & 1 & -7/5 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $AX = B$ or $LUX = B$

Let $UX = V$ then $LV = B$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & -5/3 & 0 \\ 2 & -10/3 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$3v_1 = 2 \;\; or \;\; v_1 = \frac{2}{3}$$

$$v_1 - \frac{5}{3}\,v_2 = 3 \Rightarrow v_2 = \left(3 - \frac{2}{3}\right)\left(\frac{-3}{5}\right) = \frac{-7}{5}$$

$$2v_1 - \frac{10}{3}\,v_2 = 5v_3 = 1 \Rightarrow v_3 = \frac{-1}{5}\left[1 - \frac{4}{3} - \frac{14}{3}\right] = 1$$

$$\therefore \; V = \begin{bmatrix} 2/3 \\ -7/5 \\ 1 \end{bmatrix}$$

$$UX = V$$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{5}{3} \\ 0 & 1 & -7/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/3 \\ -7/5 \\ 1 \end{bmatrix}$$

By back substitution

$$z = 1$$

$$y - \frac{7}{5} z = \frac{-7}{5} \quad \text{or} \quad y = 0$$

$$x - \frac{1}{3} y + \frac{5}{3} z = \frac{2}{3} \Rightarrow x = \frac{2}{3} \quad \frac{-5}{3} = -1$$

$\therefore$ The solution of the given equations is

$$\boxed{x = -1, \ y = 0, \ z = 1}.$$

14 (b) Using iteration method, find the dominant eigen value and eigen vector of the matrix $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.

Sol:- Take the initial vector $X^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$AX^{(0)} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.107 \end{bmatrix} = 6x_1 \lambda^{(1)} X^{(1)}$$

$$AX^{(1)}=\begin{bmatrix}6 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.107\end{bmatrix}=\begin{bmatrix}6.834\\1,334\end{bmatrix}=6.834\begin{bmatrix}1\\0.199\end{bmatrix}$$

$$\begin{bmatrix}6 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.195\end{bmatrix}=\begin{bmatrix}6.975\\1.39\end{bmatrix}=6.975\begin{bmatrix}1\\0.199\end{bmatrix}$$

$$\begin{bmatrix}6 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.199\end{bmatrix}=\begin{bmatrix}6.995\\1.398\end{bmatrix}=6.995\begin{bmatrix}1\\0.199\end{bmatrix}$$

The largest (dominant) given value is 6.995 $\cong$7 and the corresponding given vector is

$$\boxed{\begin{bmatrix}1\\0.199\end{bmatrix}\cong\begin{bmatrix}1\\0.2\end{bmatrix}}.$$

15. a) Solve : $\left(y+2x^3\right)\dfrac{dx}{dy}=x$

Sol :- Given that

$$\left(y+2x^3\right)\frac{dx}{dy}=x$$

$$\left(\frac{y+2x^3}{x}\right)=\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{y}{x}+\frac{2x^3}{x}$$

$$\frac{dy}{dx}-\frac{y}{x}=2x^2 \qquad \text{------------} (1)$$

Which is Leibnitz linear equation.

$$I.F = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}.$$

The solution is given by $y.\frac{1}{x} = \int 2x^2.\frac{1}{x} \; dx + C$

or $\frac{y}{x} = x^2 + C$

or $\boxed{y = x^3 + Cx}$

15. **b)** $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

Sol :- Given that $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

Above equation is homogenous equation

Hence

put $y = Vx$

$$\frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$V + x\frac{dV}{dx} = \frac{V}{1+\sqrt{V}}$$

$$x\frac{dV}{dx} = \frac{V}{1+\sqrt{V}} - V$$

$$x\frac{dV}{dx} = \frac{\cancel{V} - \cancel{V} - V\sqrt{V}}{1+\sqrt{V}}$$

$$x\frac{dV}{dx} = \frac{-V\sqrt{V}}{1+\sqrt{V}}$$

$$dV\left[\frac{1+\sqrt{V}}{V\sqrt{V}}\right] = -\frac{dx}{x}$$

Integrating both sides

$$\int\left[\frac{1}{V^{3/2}} + \frac{1}{V}\right]dV = -\log x + C$$

$$\int\frac{1}{V^{3/2}}\,dV + \int\frac{1}{2}dV = -\log x + C$$

$$\frac{V^{3/2+1}}{-3/2+1} + \log V = -\log x + C$$

$$-2\left[\frac{1}{\sqrt{V}}\right] + \log\left(\frac{y}{x}\right) + \log x = C$$

$$\boxed{\log y - 2\sqrt{\frac{x}{y}} = C}$$

15. (c) Solve $y + px = x^4 p^2$

Sol:- Differentiating the given equation with respect to x

$$\frac{dy}{dx} + p + x\frac{dp}{dx} = 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

$$x(1-2x^3p)\frac{dp}{dx} + 2p(1-2x^3p) = 0$$

$$(1-2x^3p)\left(x\frac{dp}{dx} + 2p\right) = 0$$

$$\Rightarrow \quad 1-2x^3p = 0 \quad or \quad x\frac{dp}{dx} + 2p = 0$$

$1-2x^3p = 0$ gives singular solution (with out constant) hence neglected.

$$x\frac{dp}{dx} + 2p = 0 \quad or \quad \frac{dx}{x} = \frac{dp}{-2p}$$

Integrating bothsides

$$log\, x = -\frac{1}{2}\, log\, p + log\, c$$

or $\quad x^2p = c, \quad or \quad p = \frac{c}{x^2}$

Substituting in the given equation

$$y + \frac{cx}{x^2} = \frac{x^4c^2}{x^4}$$

or $$\boxed{y = c^2 - \frac{c}{x}}$$

16. (a) Solve $(2x^2 + y^2 + x)\,dx + xy\,dy = 0,\ y(1) = 1$ **.It is of the form** $M\,dx + N\,dy = 0$

Sol:- $\frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = y,\ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - y}{xy} = \frac{1}{x}$$

$\therefore$ Integrating factor $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Multiplying through out by x

$x(2x^2 + y^2 + x)\,dx + x^2 y = 0$ ---------(1)

Now $\frac{\partial M}{\partial y} = 2xy = \frac{\partial N}{\partial x}$

$\therefore$ Eq. (1) is exact

Its solution is given by

$$\int_{y=const} (2x^3 + xy^2 + x^2)\,dx = c$$

$$\Rightarrow \frac{2x^4}{4} + \frac{x^2 y^2}{2} + \frac{x^3}{3} = c$$

given y(1) =1 $\Rightarrow \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = c \quad or \quad c = \frac{4}{3}$

$\therefore$ The solution is

$$\boxed{\frac{x^4}{2} + \frac{x^2 y^2}{2} + \frac{x^3}{3} = \frac{4}{3}}$$

16. b) Find the Orthogonal Trajectories of the cardiods

$r = a(1+\cos\theta)$

Sol :- Given that

$$r = a(1+\cos\theta) \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$

$$\frac{dr}{d\theta} = a\,(-\sin\theta) \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

Eliminating 'a' from (1) and (2)

$$\frac{dr}{d\theta} = \frac{-\sin\theta . r}{1+\cos\theta}$$

The differential equation of the orthogonal trajectory is given

by replacing $\frac{dr}{d\theta}$ $by\ -r^2 \frac{d\theta}{dr}$

$$-r^2 \frac{d\theta}{dr} = \frac{-r\sin\theta}{1+\cos\theta}$$

or $$r\frac{d\theta}{dr} = \frac{\sin\theta}{1+\cos\theta}$$

or $$\frac{dr}{r} = \frac{1+\cos\theta}{\sin\theta} d\theta$$

Integrating both sides

$$\log r = \int \frac{1+\cos\theta}{\sin\theta}\, d\theta + \log C$$

$$= \int \frac{2\cos^2 \theta/2}{2\sin\theta/2 \cos\theta/2}\, d\theta + \log C$$

$$= \int \frac{\cos\theta/2}{\sin\theta/2}\, d\theta + \log C$$

$$= 2\ \log\sin\theta/2 + \log C$$

$$= \log\frac{(1-\cos\theta)}{2} + \log C$$

$$= \log\ C(1-\cos\theta)$$

or $$\boxed{r = C(1-\cos\theta)}$$

17. a) Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{-2x} + x^2 + 1$

Sol :- Given that

$$(D^2+4D+4)y = e^{-2x} + x^2 + 1$$

Its Auxiliary equation is $m^2+4m+4=0$

or $(m+2)^2 = 0$

Its roots are -2, -2.

$$\therefore\ y_C = (C_1 + C_2 x)e^{-2x} \ \underline{\qquad\qquad} (1)$$

$$y_p = \frac{e^{-2x} + x^2 + 1}{D^2 + 4D + 4}$$

$$y_p = \frac{e^{-2x}}{D^2 + 4D + 4} + \frac{1 + x^2}{D^2 + 4D + 4}$$

$$y_p = y_{p_1} + y_{p_2}$$

$$y_{p_1} = \frac{e^{-2x}}{D^2 + 4D + 4} = \frac{x^2\, e^{-2x}}{2}$$

$$y_{p_2} = \frac{1 + x^2}{D^2 + 4D + 4}$$

$$= \frac{1}{4}\left[\frac{1}{\left(\frac{D^2 + 4D}{4}\right) + 1}\right](1 + x^2)$$

$$= \frac{1}{4}\left[1 + \left(D + D^2/4\right)\right]^{-1}(1 + x^2)$$

$$\left[\because (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 \ldots\right]$$

$$= \frac{1}{4}\left[1 - \left(D + \frac{D^2}{4}\right) + \left(D + \frac{D^2}{4}\right)^2\right](1 + x^2)$$

$$= \frac{1}{4}\left[1 - \left(D + \frac{D^2}{4}\right) + (D^2)\right](1 + x^2)$$

(neglecting higher derivatives after second order)

$$= \frac{1}{4}\left(1 - D + \frac{3}{4}D^2\right)\left(1 + x^2\right)$$

$$= \frac{1}{4}\left(1 + x^2 - 2x + \frac{3}{4}.2\right)$$

$$= \frac{1}{4}\left(x^2 - 2x + \frac{5}{2}\right)$$

$$y_p = \frac{x^2 e^{-2x}}{2} + \frac{1}{4}\left[x^2 - 2x + \frac{5}{2}\right]. \qquad \text{\_\_\_\_\_\_(2)}$$

The complete solution is $y = y_c + y_p$

$$\boxed{y = \left(C_1 + C_2 x\right)e^{-2x} + \frac{x^2 e^{-2x}}{2} + \frac{1}{4}\left(x^2 - 2x + \frac{5}{2}\right)}$$

17 . (b) Solve $\frac{dx}{dt} = 2x - 3y$; $\frac{dy}{dt} = 3x + y$

Sol:- The given equation can be written as

$(D-2)\,x + 3y = 0$ -------(1)

$-3x + (D-1)\,y = 0$ -------(2)

$(1) \times 3 + (2) \times (D-2)$ gives

$[(D-1)\ (D-2) + 9]\ y = 0$

$$(D^2 - 3D + 11)\ y = 0$$

Its auxillary equation is $m^2 - 3m + 11 = 0$ and the roots are

$$\frac{3 \pm \sqrt{9-44}}{2} = \frac{3 \pm \sqrt{35}i}{2}$$

$$\therefore\ y = e^{\frac{3}{2}t}\left[C_1\ Cos\ \frac{\sqrt{35}}{2}t + C_2\ Sin\ \frac{\sqrt{35}}{2}t \right] \qquad \text{-----(3)}$$

$$3x = \frac{dx}{dt} - y$$

$$= \frac{3}{2}\ e^{\frac{3t}{2}}\left[-\frac{\sqrt{35}}{2}\ C_1\ Sin\ \frac{\sqrt{35}}{2}t + C_2 \frac{\sqrt{35}}{2}\ Cos\ \frac{\sqrt{35}}{2}t \right]$$

$$+\ e^{\frac{3t}{2}}\left[-\frac{\sqrt{35}}{2}\ C_1\ Sin\ \frac{\sqrt{35}}{2}t + C_2 \frac{\sqrt{35}}{2}\ Cos\ \frac{\sqrt{35}}{2}t \right]$$

$$= e^{\frac{3t}{2}}\left[\frac{(3+\sqrt{35})}{2}\ C_1\ Cos\ \frac{\sqrt{35}}{2}t + \frac{(3-\sqrt{35})}{2}\ C_2\ Sin\frac{\sqrt{35}}{2}t \right]$$

$$\therefore x = e^{\frac{3t}{2}}\left[\left(\frac{3+\sqrt{35}}{6} \right) C_1\, Cos\ \frac{\sqrt{35}}{2}t + \left(\frac{3-\sqrt{35}}{6} \right) C_2\, Sin \frac{\sqrt{35}}{2}t \right]$$

-------(4)

(3) and (4) constitute the solution of the given simultaneous equations.

18. a) Solve : $\frac{x^2 d^2 y}{dx^2} + \frac{2xdy}{dx} + 6y = (1+x)^2$

Sol :- Given that

$$x^2 D^2 y + 2xDy + 6y = (1+x)^2 \quad \text{\_\_\_\_\_(1)}$$

Put $e^Z = x$ *or* $z = \log x$

Then $xD = \Theta$

$$x^2 D^2 = \Theta(\Theta - 1)$$

Then $\left[x^2 D^2 + 2xD + 6\right] y = (1+x)^2$ becomes

$$\{\Theta(\Theta-1) + 2\Theta + 6\} y = (1+e^z)^2$$

or $$\left(\Theta^2 + \Theta + 6\right) y = e^{2z} + 2e^z + 1 \quad \text{\_\_\_\_\_(2)}$$

Its auxiliary equation is $m^2 + m + 6 = 0$

Its roots are $\frac{-1 \pm \sqrt{1-24}}{2} = \frac{-1 \pm \sqrt{23}\, i}{2}$

The complete solution of (2) is

$$y = C.F + P.I \quad \text{\_\_\_\_\_(3)}$$

$$C.F = e^{z/2}\left[C_1 \cos\frac{\sqrt{23}}{2} z + C_2 \sin\frac{\sqrt{23}}{2} z\right]$$

$$P.I = \frac{e^{2z} + 2e^{z} + 1}{\Theta^2 + \Theta + 6}$$

$$\frac{e^{2z}}{\Theta^2 + \Theta + 6} = \frac{e^{2z}}{2^2 + 2 + 6} = \frac{e^{2z}}{12}$$

$$\frac{2e^{z}}{\Theta^2 + \Theta + 6} = 2.\frac{e^{z}}{1 + 1 + 6} = \frac{e^{z}}{4}$$

$$\frac{1}{\Theta^2 + \Theta + 6} = \frac{1}{6}$$

$$\therefore \quad P.I = \frac{1}{12}\left(e^{2z} + 3e^{z} + 2\right)$$

Then (3) can be written as

$$y = e^{z/2}\left[C_1 \cos\frac{\sqrt{23}}{2}\, z + C_2 \sin\frac{\sqrt{23}z}{2}\right]$$

$$+ \frac{1}{12}\left(e^{2z} + 3e^{z} + 2\right)$$

Replacing z by $\log\ x$, the complete solution of (1) is

$$y = \sqrt{x}\left[C_1 \cos\frac{\sqrt{23}}{2}(\log x) + C_2 \sin\frac{\sqrt{23}}{2}(\log x)\right]$$

$$+ \frac{1}{12}\left(x^2 + 3x + 2\right).$$

18. b) **Solve** $\frac{d^2y}{dx^2} + y = x \sin x$

Sol :- The given equation in symbolic form is $\left(D^2+1\right) y = x \sin x$

Its auxiliary equation is $m^2+1=0$, its roots are $\pm\, i$

Complimentary function $\left(C.F\right) = C_1 \cos x + C_2 \sin x$

$$P.I = \frac{x \sin x}{D^2+1}$$

$$= \left[x - \frac{2D}{D^2+1}\right] \frac{\sin x}{D^2+1}$$

$$= \left(x - \frac{2D}{D^2+1}\right)\left(\frac{-x\cos x}{2}\right)$$

$$= \frac{-x^2}{2} \cos x + \frac{D}{D^2+1} \left(x \cos x\right)$$

$$= \frac{-x^2}{2} \cos x + \frac{1}{D^2+1} \left(\cos x - x \sin x\right)$$

$$= \frac{-x^2}{2} \cos x + \frac{\cos x}{D^2+1} - \frac{x \sin x}{D^2+1}$$

$$\frac{x \sin x}{D^2+1} = \frac{1}{2}\left[\frac{-x^2}{2} \cos x + \frac{x \sin x}{2}\right]$$

$$\left(\because \frac{\cos x}{D^2+1} = \frac{x\sin x}{2}\right)$$

The complete solution is

$$y = C.F + P.I$$

i.e., $$\boxed{y = C_1 \cos x + C_2 \sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x.}$$

19. a) Find the Laplace transform of the following

i) $f(t) = t^2 \cos at$

Sol :- $$L\left(t^2 \cos at\right) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left|\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right.$$

$$= \frac{d}{ds}\left[\frac{\left(s^2+a^2\right)1 - s(2s)}{\left(s^2+a^2\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right]$$

$$= \frac{\left(s^2+a^2\right)^2(-2s) - \left[\left(a^2-s^2\right).2\left(s^2+a^2\right).2s\right]}{\left(s^2+a^2\right)^4}$$

$$= \frac{(s^2+a^2)(-2s)-4s(a^2-s^2)}{(s^2+a^2)^3}$$

$$= \frac{2s^3-6a^2s}{(s^2+a^2)^3}$$

ii) $f(t) = \frac{1-\cos h\,at}{t}$

We know that $L\left(\frac{\phi(t)}{t}\right) = \int_s^\infty \bar{\phi}(s)\,ds$

Where $\phi(t) = 1-\cosh at$

$$\therefore L\left[\frac{1-\cos h\,at}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2-a^2}\right)$$

$$= \left[\log s - \frac{1}{2}\log(s^2-a^2)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2-a^2}}\right]_s^\infty$$

$$= \left[\log \sqrt{\frac{s^2}{s^2-a^2}}\right]_s^\infty = 0 - \log \sqrt{\frac{s^2}{s^2-a^2}}$$

$$= \log \sqrt{\frac{s^2 - a^2}{s^2}}$$

$$= \boxed{\frac{1}{2} \log\left(1 - \frac{a^2}{s^2}\right)}.$$

19. (b) Find the inverse transform of

i) $\cot^{-1}(s+1)$.

Sol:- Given $\overline{f}(s) = \cot^{-1}(s+1)$

We know that $L(t\, f(t)) = -\frac{d}{ds}\left[\overline{f}(s)\right]$

or $\quad t\, f(t) = L^{-1}\left[-\frac{d}{ds}\left(\overline{f}(s)\right)\right]$

$$t\, f(t) = L^{-1}\left[-\frac{d}{ds} \cot^{-1}(s+1)\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)^2 + 1}\right]$$

$$= e^{-t}.\sin t$$

$$\therefore \boxed{f(t) = e^{-t} \frac{\sin t}{t}}.$$

19. (b) (ii) Find $L^{-1}\left(\frac{1}{s^3-a^3}\right)$

Sol:- $$\frac{1}{s^3-a^3}=\frac{1}{(s-a)(s^2+as+a^2)}$$

$$=\frac{A}{s-a}+\frac{Bs+C}{s^2+as+a^2}$$

$$1 = A(s^2 + as + a^2) + (Bs+C)(s-a)$$

Equating the coefficients of 's' on both sides we get

$$\text{A} = - \ B=\frac{1}{3a^2},\ C=-\frac{2}{3a}$$

$$L^{-1}\left(\frac{1}{s^3-a^3}\right)=L^{-1}\left(\frac{1}{3a^2}\ \frac{1}{\text{s}-\text{a}}\right)-\frac{1}{3a^2}L^{-1}\left(\frac{s+2a}{s^2+as+a^2}\right)$$

$$= L^{-1}\left(\frac{1}{s^3-a^3}\right)=L^{-1}\left(\frac{1}{3a^2}\ \frac{1}{s-a}\right)-\frac{1}{3a^2}\ L^{-1}\left(\frac{s+2a}{s^2+as+a^2}\right)$$

$$= \frac{1}{3a^2}\ e^{at}-\frac{1}{3a^2}\ e^{-1}\left[\frac{\left(s+\frac{a}{2}\right)+\frac{3}{2}a}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}}{2}a\right)^2}\right]$$

$$= \boxed{\frac{1}{3a^2}\ e^{at}-\frac{1}{3a^2}\left(e^{-at/2}\ \cos\frac{\sqrt{3}}{2}\ at+\sqrt{3}\,e^{-at/2}\sin\frac{\sqrt{3}}{2}at\right)}$$

20. (a) Find the laplace transform of the following periodic function

$$f(t) = \begin{cases} t \quad , \quad 0<t<a \\ 2a-t \, , \quad a<t<2a \end{cases} \quad \text{and} \quad f(t+2a) = f(t).$$

Sol:- $$L[f(t)] = \frac{1}{1-e^{-st}} \int_O^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_o^a t e^{-st} dt + \int_a^{2a} (2a-t)\, e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right)_0^a + \left((2a-t) \frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right)_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(\frac{-a e^{-st}}{s} - \frac{-e^{-as}}{s^2} + \frac{1}{s^2} \right) + \left(\frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} \right) \right]$$

$$= \frac{1}{s^2 \left(1-e^{-2as}\right)} \left(e^{-2as} - 2e^{-as} + 1\right)$$

$$= \frac{1}{s^2} \frac{\left(1-e^{-as}\right)^2}{1-e^{-2as}} = \frac{1}{s^2} \frac{\left(1-e^{-as}\right)}{\left(1+e^{-as}\right)}$$

$$= \frac{1}{s^2} \frac{\left(e^{as/2} - e^{-as/2}\right)}{\left(e^{as/2} + e^{-as/2}\right)} = \boxed{\frac{1}{s^2} \tan h \frac{as}{2}}$$

20. (b) Find $L\left[t^2 u(t-3)\right]$

Sol:- f (t) = t^2

express f(t) as a function of (t - 3)

$t^2 = (t-3)^2 + 6t + 9 = (t-3)^2 + 6(t-3) + 18 - 9$

$= (t-3)^2 + 6(t-3) + 9$

$$L\left[t^2 u(t-3)\right] = \left[(t-3)^2 + 6(t-3) + 9\right] u(t-3)$$

$$= \frac{2}{s^3} e^{-3s} + \frac{6}{s^2} e^{-3s} + \frac{9}{s} e^{-3s}$$

(using second shifting property)

$$\boxed{L\left[t^2 u(t-3)\right] = \frac{e^{-3s}}{s^3}\left(2s^2 + 6s + 9\right)}$$

20. (c) Solve $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(o) = 1$ **using laplace transiforms.**

Sol:- Applying laplace transform bothsides of the given equation, we get.

$$s\overline{y}(s) - y(o) + 2\overline{y}(s) + \frac{\overline{y}(s)}{s} = \frac{1}{s^2+1}$$

or $\overline{y}(s)\left(s + 2 + \frac{1}{s}\right) = \frac{1}{s^2+1} + 1$

$$\bar{y}(s)\frac{s}{\left(s^2+1\right)\ \left(s^2+2s+1\right)}+\frac{s}{s^2+2s+1}$$

splitting this into partial fractions

$$\bar{y}(s)\frac{1}{2\left(s^2+1\right)}-\frac{3}{2\left(s+1\right)^2}+\frac{1}{s+1}$$

taking inverse Laplace transform bothsides, we get

$$\boxed{y(t)=\frac{1}{2}\sin t-\frac{3}{2}e^{-t}t+e^{-t}}$$

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